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Topological strings and large N phase transitions II: chiral expansion of q-deformed Yang-Mills theory

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Abstract: We continue our study of the large N phase transition in q-deformed Yang-Mills theory on the sphere and its role in connecting topological strings to black hole entropy. We study in detail the chiral theory defined in terms of uncoupled single $\mathrm{U}(N)$ representations at large N and write down the resulting partition function by means of the topological vertex. The emergent toric geometry has three Kähler parameters, one of which corresponds to the expected fibration over $\mathbb{P}^1$. By taking a suitable double-scaling limit we recover the chiral Gross-Taylor string expansion. To analyse the phase transition we construct a matrix model which describes the chiral gauge theory. It has three distinct phases, one of which should be described by the closed topological string expansion. We verify this expectation by explicit comparison between the matrix model and the chiral topological string free energies. We also show that the critical point in the pertinent phase of the matrix model corresponds to a divergence of the topological string perturbation series.

Keywords: Topological Strings, Field Theories in Lower Dimensions, Nonperturbative Effects, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

The study of black holes in string theory has received a renewed attention after the Ooguri-Strominger-Vafa conjecture [1] which proposes a connection between the entropy of four dimensional BPS black holes and topological string amplitudes. A possible way to construct such a black hole is through a Calabi-Yau compactification of Type IIA superstring theory which yields $\mathcal{N} = 2$ supergravity in four dimensions. Four dimensional black hole microstates are interpreted as bound states of D-branes wrapping cycles of the internal manifold and the microscopic entropy agrees at leading order with the Bekenstein-Hawking entropy computed in the corresponding effective field theory [2]. Rather remarkably, this relation holds beyond leading order when we include corrections to the Bekenstein-Hawking area law that arise from higher derivative F-term interactions in the effective theory [3]–[5] (see [6] for a review).

On the other hand, the physics of the F-term in the four dimensional effective field theory is captured by a twisted sigma-model defined on the Calabi-Yau manifold. The genus zero free energy computes the effective action for the vector multiplets (in the Type IIA setting) up to two derivatives, while higher genus topological string amplitudes encode information about higher derivative couplings between the curvature and the graviphoton field strength [7, 8] (see [9]–[12] for reviews).

From this relation it have been argued [1] that the partition function of the black hole at the attractor point [13, 14] is equal to the modulus squared of the topological string vacuum amplitude, $Z_{\rm BH} = |Z_{\rm top}|^2$. This conjecture has been addressed and refined in [15]–[26]. A concrete realization of this proposal has been advanced in [27] and subsequently in [28] for local threefolds X_p which are fibred over a compact curve Σ_g of genus g. In this case a black hole can be engineered as bound states of N D4-branes wrapping a non-compact four-cycle, and an arbitrary number of D-branes wrapping Σ_g along with D0-branes. This configuration realizes a four dimensional BPS black hole whose microstates are described by a mixed ensemble of fixed magnetic D4-brane charge N and electric chemical potentials. Bound states of D-branes are counted by the corresponding gauge field configurations excited on the worldvolume of the D4-brane. These in turn localize to a deformed version of $\mathrm{U}(N)$ Yang-Mills theory on the Riemann surface Σ_g, the q-deformed Yang-Mills theory introduced in [29, 28]. In the large N limit, this theory factorizes into a chiral and antichiral part, like its undeformed cousin QCD_2 [30]–[32], corresponding to the holomorphic and antiholomorphic structure of $|Z_{\rm top}|^2$. The topological string amplitude $Z_{\rm top}$ on these geometries has been computed recently in [33].

In the following we will focus on the case where the local threefold X_p is fibered over a two-sphere. This paper is a companion of [34] where the gauge theoretical aspects of the correspondence were analysed. Here we will study in detail the emergence of the topological string theory from the strong coupling phase of the chiral two-dimensional q-deformed gauge theory at large N. A crucial outcome of our investigation is a path towards a topological sigma-model description underlying the Gross-Taylor string expansion [30]–[32] of ordinary Yang-Mills theory. While at zero coupling (where QCD_2 is a topological field theory) such a reformulation exists [35], at finite area the full description in terms of topological string theory is problematic. We present a potential way to overcome these difficulties which relies on viewing QCD_2 as a particular limit of the q-deformed gauge theory. As we will show, the latter model admits a well-defined interpretation as a topological string theory from which one can in principle extract the precise string theory underlying QCD_2. Moreover, as the topological strings live in higher target space dimensions this approach has the potential of extending the two-dimensional Gross-Taylor string description to QCD_4.

Many of the features of two-dimensional Yang-Mills theory on a sphere, such as its non-trivial instanton driven phase transition in the large N limit [36, 37], are preserved by the q-deformation and were thoroughly addressed in [34, 38, 39]. In this paper we will focus on the chiral-antichiral decomposition of q-deformed Yang-Mills theory, which according to [27, 28] is the chiral (antichiral) sector that should be related to the holomorphic (antiholomorphic) topological string amplitudes. Similarly to what has been observed in [40, 41] for the undeformed theory, we will show that the chiral sector has a non-trivial phase structure that plays a key role in the comparison to string theory.

On the other hand, the threefold X_p is one of the simplest and best studied examples of a toric variety. The topological vertex [42] is a useful tool for computing topological string amplitudes in these geometries. A toric threefold is characterized by a graph where trivalent vertices are glued together and the edges signal the degeneration of a $\mathbb{T}^2$ fibration. Topological string amplitudes can be reduced (roughly by an appropriate placing of brane-antibrane pairs) to those on patches of trivial topology. The topological vertex is the building block of this construction. We will use this formalism to make explicit contact with the chiral q-deformed gauge theory at large N. In this setup we can make a remarkable check of the proposal by exhibiting a relation between the Hurwitz numbers that compute the combinatorics of branched covering maps to $\mathbb{P}^1$ and the relevant Gromov-Witten invariants that "count" in the appropriate way the holomorphic curves embedded in the Calabi-Yau manifold. Moreover, the string theory exhibits its origin as a gauge theory through the finiteness of the radius of convergence of its perturbative expansion.

This paper is organized as follows. Section 2 is devoted to exploring the relationship between the large N chiral q-deformed gauge theory partition function and the closed topological string amplitude on the local threefold X_p. We rewrite the free energy of the chiral theory in the large N limit in terms of (generalized) Gromov-Witten invariants and show that its undeformed limit agrees perfectly with the conventional chiral QCD_2 result. We rewrite the chiral partition function in terms of the topological vertex as a topological string amplitude and analyse the Gromov-Witten invariants of the corresponding toric scheme $\hat{X}_p$. As expected, the related Gopakumar-Vafa invariants of $\hat{X}_p$ are integral for all p. We also use the undeformed limit to derive an asymptotic localization formula for the Gromov-Witten invariants in terms of Hodge integrals.

Section 3 is devoted to the analysis of the large N phase structure of the chiral gauge theory by means of a matrix model, and the recovery of the topological string theory in the strong coupling phase. Our saddle-point equations consistently reduce to the Crescimanno-Taylor equations [41] in the undeformed limit and we find a qualitatively similar phase diagram to that of chiral QCD_2. In particular, two phase transitions occur and we show that the one-cut solution in the strong coupling phase agrees with the expectations from topological string theory.

Finally, in section 4 we examine the analytic properties of the perturbative expansion of the string partition function. We find a finite radius of convergence that corresponds to the critical points of the phase transition. We compare the value of the critical point obtained in this way with both the numerical results of the matrix model analysis and with the exact result for the full coupled gauge theory found in [34]. We also speculate that the finite radius of convergence of the topological string amplitude may have an interpretation as a sort of Hagedorn transition. Section 5 contains some concluding remarks and we collect various technical details in four appendices at the end of the paper.

2. q-deformed Yang-Mills theory on S^2 and closed topological strings

The conjectured relation in [1] links four-dimensional black holes with topological strings. Consider Type IIA string theory on the local Calabi-Yau threefold

$$X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \ \longrightarrow \ \mathbb{P}^1 \ , \tag{2.1}$$

where $p \in \mathbb{Z}$ and $\mathcal{O}(m)$ is the canonical holomorphic line bundle over $\mathbb{P}^1$ of degree m. The conjecture predicts that the black hole partition function $Z_{\rm BH}$ is related for large charges to the perturbative vacuum amplitude $Z_{\rm top}$ for topological strings on X_p by $Z_{\rm BH} = |Z_{\rm top}|^2$. Because an exact microscopic computation [28] shows that $Z_{\rm BH}$ coincides with the partition function $\mathcal{Z}^q_{\rm YM}$ of q-deformed $\mathrm{U}(N)$ Yang-Mills theory on the base $S^2 \cong \mathbb{P}^1$ of the fibration (2.1), a natural check of the conjecture is to verify that $\mathcal{Z}^q_{\rm YM} = Z_{\rm top}\,\overline{Z}_{\rm top}$ in the large N limit. In this section we shall analyse this last relation by considering in detail the large N *chiral* expansion of the q-deformed gauge theory. We will find the deformed analog of chiral QCD_2 and exhibit its precise relation with the closed topological string amplitude on the Calabi-Yau threefold X_p. Unless explicitly stated otherwise, we will assume that $p > 2$ as this is when the large N phase transition occurs.

2.1 Large N expansion

The partition function of q-deformed Yang-Mills theory on the sphere S^2 is given by

$$\mathcal{Z}^q_{\rm YM} = \sum_R \dim_q(R)^2 \ q^{\frac{p}{2}\,C_2(R)} \ , \tag{2.2}$$

where the sum runs over all irreducible representations of the $\mathrm{U}(N)$ gauge group, $q = \mathrm{e}^{\,-g_s}$, $C_2(R)$ is the quadratic Casimir invariant of R, and the quantum dimension is defined as

$$\dim_q(R) = \prod_{1\le i<j\le N} \frac{\big[R_i - R_j + j - i\big]_q}{\big[j-i\big]_q} = \prod_{1\le i<j\le N} \frac{q^{(R_i-R_j+j-i)/2} - q^{-(R_i-R_j+j-i)/2}}{q^{(j-i)/2} - q^{-(j-i)/2}} \tag{2.3}$$

with R_i labeling the number of boxes in the i-th row of the Young tableau corresponding to R. In [28] a different but related definition is used whereby the quantum dimension is replaced by the quantity $S_{00}\dim_q(R)$ with

$$S_{00}(q,N) = \prod_{1\le i<j\le N} [j-i]_q \ . \tag{2.4}$$

At finite N the presence of S_{00} is simply a change in the overall normalization. In the large N limit it will produce the contribution of constant maps to the topological string amplitude [44, 49], a universal factor that can be computed separately (see appendix A). We will include this contribution later on when we compare our results with topological string theory.

In [28] the asymptotic expansion in $\frac{1}{N}$ of the $\mathrm{U}(N)$ partition function (2.2) was constructed following closely the strategy proposed in [30]–[31] for ordinary two-dimensional Yang-Mills theory. The sum was restricted to a subset of representations called "composite" large N representations. These are essentially the representations whose quadratic Casimir invariant has a leading term of order N. Composite representations are formed by taking the tensor product of a representation R corresponding to a Young diagram with a finite number of boxes and a representation $\bar{S}$ which is the complex conjugate of a representation S associated to another diagram with a finite number of boxes. The resulting large N expansion essentially factorizes into two copies of a simpler chiral topological string expansion, but with a couple of important subtleties. Firstly, one should include in the definition of $Z_{\rm top}$ a sum over a $\mathrm{U}(1)$ degree of freedom identified with a Ramond-Ramond flux through the sphere. Secondly, and more importantly, the relevant topological string partition function implies the presence of two stacks of D-branes inserted in the fibers of X_p, represented by two extra sums over representations with finite numbers of boxes.

The explicit result obtained in [28] reads

$$\mathcal{Z}^q_{\rm YM} = \sum_{l=-\infty}^{\infty} \ \sum_{\hat{R}_{(1)},\hat{R}_{(2)}} \mathcal{Z}^{q{\rm YM},+}_{\hat{R}_{(1)},\hat{R}_{(2)}}(t+p\,g_s l;p)\, \mathcal{Z}^{q{\rm YM},-}_{\hat{R}_{(1)},\hat{R}_{(2)}}(\,\bar{t}-p\,g_s l;p) \tag{2.5}$$

where the second sum runs through irreducible representations $\hat{R}_{(1)},\hat{R}_{(2)}$ of $\mathrm{SU}(N)$ with

$$\mathcal{Z}^{q{\rm YM},-}_{\hat{R}_{(1)},\hat{R}_{(2)}}(\,\bar{t};p) = (-1)^{|\hat{R}_{(1)}|+|\hat{R}_{(2)}|}\, \mathcal{Z}^{q{\rm YM},+}_{\hat{R}^\top_{(1)},\hat{R}^\top_{(2)}}(\,\bar{t};p)\ . \tag{2.6}$$

The Kähler modulus t parameterizes the area of the sphere $\mathbb{P}^1$ and is given by

$$t = (p-2)\,\frac{N\,g_s}{2} + {\rm i}\,\theta \tag{2.7}$$

in terms of the original parameters of the gauge theory (in this paper we set $\theta=0$). The symbol $|\hat{R}|$ is the total number of boxes of the Young tableau of the $\mathrm{SU}(N)$ representation $\hat{R}$. The chiral block $\mathcal{Z}^{q{\rm YM},+}_{\hat{R}_{(1)},\hat{R}_{(2)}}(t;p)$ agrees exactly with the perturbative topological string amplitude on X_p [33] with two stacks of D-branes inserted in the fiber. It depends explicitly on the choice of two arbitrary Young tableaux which correspond to the boundary degrees of freedom of the fiber D-branes. When all the Young tableaux are taken to be trivial, i.e. $\hat{R}_{(1)}=\hat{R}_{(2)}=0$, one recovers the expected closed topological string partition function. The chiral and anti-chiral parts are sewn together along the D-branes and summed over them.

The full non-chiral partition function also admits a standard description in terms of toric geometry. As we show explicitly in section 2.3, the fibration (2.1) is a toric manifold and the chiral block $\mathcal{Z}^{q{\rm YM},+}_{\hat{R}_{(1)},\hat{R}_{(2)}}(t;p)$ can be written in terms of the topological vertex $C_{\hat{R}_{(1)}\hat{R}_{(2)}\hat{R}_{(3)}}(q)$ [42] as

$$\begin{aligned}\mathcal{Z}^{q{\rm YM},+}_{\hat{R}_{(1)},\hat{R}_{(2)}}(t;p) &= \mathsf{Z}_0(q)\; q^{\kappa_{\hat{R}_{(1)}}/2}\; {\rm e}^{-\frac{t(|\hat{R}_{(1)}|+|\hat{R}_{(2)}|)}{p-2}} \\ &\quad\times \sum_{\hat{R}} {\rm e}^{-t|\hat{R}|}\; q^{(p-1)\kappa_{\hat{R}_{(1)}}/2}\; C_{0\hat{R}_{(1)}\hat{R}^\top}(q)\, C_{0\hat{R}\hat{R}_{(2)}}(q)\ ,\end{aligned} \tag{2.8}$$

where $\kappa_{\hat{R}}$ is related to the Young tableau labels through

$$\kappa_{\hat{R}} = \sum_{i=1}^{N-1} \hat{R}_i \big(\hat{R}_i - 2i + 1\big) \tag{2.9}$$

and $\mathsf{Z}_0(q)$ represents the contribution from constant string maps (see appendix A). This is the partition function of the topological A-model on X_p with non-compact lagrangian D-branes inserted at two of the four lines in the web diagram. The D-branes are placed at a well-defined "distance" $t/(p-2)$ from the sphere, thereby introducing another geometrical parameter.

The extra sum over the integer l originates from the U(1) degrees of freedom contained in the original gauge group U(N) and can be interpreted as a sum over Ramond-Ramond fluxes through the sphere [27]. The sum over the fiber D-branes is instead related to the fact that the Calabi-Yau manifold X_p is non-compact and has more moduli coming from the non-compact directions [43]. Note that the sum over the "external" branes in the full partition function enters on the same footing as the sum over the topological string amplitude constituents. This "external" sum is weighted with a different Kähler parameter

$$\hat{t} = \frac{t}{p-2} = \frac{N\, g_s}{2} \ , \tag{2.10}$$

and the partition function therefore effectively depends on two parameters. The observation above suggests that $\hat{t}$ could have an interpretation as a true Kähler modulus. As we will demonstrate in the following, this follows from a different definition of the chiral gauge theory which is directly connected to the ordinary Yang-Mills one and which leads to a closed topological string theory by itself. The chiral expansion we propose arises from restricting the original sum to only those representations whose Young diagrams contain a finite number n of boxes. Coupled representations will not be considered in this paper.

2.2 Chiral expansion

We now describe the chiral expansion explicitly. The second Casimir invariant for U(N) representations R has the form

$$C_2(R) = \kappa_{\hat{R}} + N\, n - \frac{n^2}{N} + \frac{m^2}{N} \ , \tag{2.11}$$

where m is the U(1) charge

$$m = n + N\, r \tag{2.12}$$

with $r \in \mathbb{Z}$. The row labels of the U(N) representations R are related to those of SU(N) representations $\hat{R}$ by $R_i = \hat{R}_i + r$ for $i = 1, \dots, N-1$ and $R_N = r$. The partition function is a sum over the total number of boxes n, the SU(N) Young tableaux $\hat{R}$ with n boxes and the U(1) degree of freedom r, giving

$$\mathcal{Z}^{q\mathrm{YM}}_{\mathrm{chiral}} = \sum_{r=-\infty}^{\infty} \mathrm{e}^{-\frac{N\, g_s p}{2}\, r^2} \sum_{n=1}^{\infty} \sum_{\hat{R}} \dim_q\big(\hat{R}\big)^2 \, \exp\left[-\frac{N\, g_s p}{2}\left(n + \frac{\kappa_{\hat{R}}}{N}\right) - g_s p\, n\, r\right] \ . \tag{2.13}$$

Let us focus on the $r = 0$ sector of vanishing Ramond-Ramond flux through S^2, whose partition function reads

$$\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}} = \sum_{n=1}^{\infty} \sum_{\hat{R}} \dim_q\big(\hat{R}\big)^2 \, \exp\left[-\frac{N\,g_s p}{2}\left(n + \frac{\kappa_{\hat{R}}}{N}\right)\right] \ . \tag{2.14}$$

To proceed further, we need to understand the structure of the quantum dimension $\dim_q(\hat{R})$. It can be conveniently expressed as [28]

$$\dim_q\big(\hat{R}\big) = \mathrm{e}^{\,\frac{g_s N}{2}\,n}\, W_{\hat{R}}\big(q^{-1}\big) \prod_{i=1}^{c_{\hat{R}}} \prod_{j=1}^{\hat{R}_i} \left(1 - q^{j-i}\ \mathrm{e}^{\,-g_s N}\right) \ , \tag{2.15}$$

where $c_{\hat{R}}$ is the number of rows in $\hat{R}$ and $W_{\hat{R}}(q) = W_{\hat{R}0}(q)$ is related to the large N limit of the modular S-matrix of the $\mathrm{SU}(N)$ WZW model and $\mathrm{SU}(N)$ Chern-Simons gauge theory by

$$W_{\hat{R}\hat{Q}}(q) = \lim_{N\to\infty} q^{\frac{N(|\hat{R}|+|\hat{Q}|)}{2}}\, \frac{S_{\hat{R}\hat{Q}}(q,N)}{S_{00}(q,N)} \ . \tag{2.16}$$

A convenient way to parameterize the different Young diagrams $\hat{R}$ with box number n is as follows. Given a partition

$$n = n_1 + n_2 + \cdots + n_{i_{\max}} \tag{2.17}$$

of n with $n_1 \geq n_2 \geq \cdots \geq n_{i_{\max}}$, one obtains the list $d(n) = (n_1, n_2, \ldots, n_{i_{\max}})$. With $n_i = \hat{R}_i$ the number of boxes in the i-th row of $\hat{R}$, we have

$$\dim_q\big(\hat{R}\big) = \mathrm{e}^{\,-\frac{g_s N}{2}\,n}\, W_{\hat{R}}\big(q^{-1}\big) \prod_{i=1}^{i_{\max}} \prod_{j=1}^{n_i} \left(1 - q^{j-i}\ \mathrm{e}^{\,-g_s N}\right) \ . \tag{2.18}$$

By using the relation

$$W_{\hat{R}}\big(q^{-1}\big) = (-1)^n\, q^{-\kappa_{\hat{R}}/2}\, W_{\hat{R}}\big(q\big) \tag{2.19}$$

we can rewrite the chiral partition function as

$$\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}} = \sum_{n=1}^{\infty} \sum_{\hat{R}} \mathrm{e}^{\,-\frac{N\,g_s(p-2)}{2}\,n}\ \mathrm{e}^{\,-\frac{g_s(p-2)}{2}\,\kappa_{\hat{R}}}\, W_{\hat{R}}(q)^2 \prod_{i=1}^{i_{\max}} \prod_{j=1}^{n_i} \left(1 - q^{j-i}\ \mathrm{e}^{\,-g_s N}\right)^2 \ . \tag{2.20}$$

The explicit expression [46]

$$W_{\hat{R}}(q) = q^{\kappa_{\hat{R}}/4} \prod_{1\leq i<j\leq i_{\max}} \frac{[n_i - n_j + j - i]_q}{[j-i]_q} \prod_{i=1}^{i_{\max}} \prod_{k=1}^{n_i} \frac{1}{[k - i + i_{\max}]_q} \tag{2.21}$$

shows that the general structure of $W_{\hat{R}}(q)$ is of the form $q^{\alpha}/\prod_{\beta,\gamma}(1-q^{\beta})^{\gamma}$ for some $\alpha,\beta,\gamma > 0$, and the leading behaviour as $g_s \to 0$ is simply determined by the total number of boxes n as $W_{\hat{R}}(q) \simeq g_s^{-n}$.

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We are ready now to make the connection with the topological string expansion. In terms of the Kähler modulus (2.7), the chiral partition function is simply rewritten as

$$\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}} = \sum_{n=1}^{\infty} \mathrm{e}^{-t\,n} \sum_{\hat{R}} \mathrm{e}^{-\frac{g_s(p-2)}{2}\,\kappa_{\hat{R}}}\, W_{\hat{R}}(q)^2 \prod_{i=1}^{i_{\max}} \prod_{j=1}^{n_i} \Big(1 - q^{j-i}\ \mathrm{e}^{-\frac{2\,t}{p-2}}\Big)^2 \ . \qquad (2.22)$$

The string expansion is obtained by expanding the partition function in powers of the string coupling constant g_s with t fixed. Notice that this is not a power series in e^{-t} at any given order in g_s, as one would expect in a conventional topological string perturbative expansion with a single Kähler modulus. There is also a power series in $\mathrm{e}^{-\frac{2\,t}{p-2}}$ which in the coupled partition function appears as fiber D-brane contributions whose distance from the S^2 is parameterized by the Kähler modulus (2.10). We will come back to this point later on. The general structure of the free energy $\mathcal{F}^{q\mathrm{YM}}_{\mathrm{chiral}} = \log \mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}}$ is given by

$$\mathcal{F}^{q\mathrm{YM}}_{\mathrm{chiral}} = \sum_{g=0}^{\infty} g_s^{2g-2}\ \mathcal{F}_g \qquad (2.23)$$

with

$$\mathcal{F}_g\big(t\,,\,\hat{t}\,;\,p\big) = \sum_{n=1}^{\infty} \mathrm{e}^{-n\,t} \sum_{k=0}^{2n} \mathrm{e}^{-2\,k\,\hat{t}}\ \mathrm{N}^g{}_{n,k}(p)\ , \qquad (2.24)$$

where n labels the winding numbers of holomorphic maps from genus g Riemann surfaces into the local threefold X_p and $\mathrm{N}^g{}_{n,k}(p) \in \mathbb{Q}$ are generalized Gromov-Witten invariants. The conventional Gromov-Witten invariants $\mathrm{N}^g{}_n = \mathrm{N}^g{}_{n,k=0}$ of the local Calabi-Yau threefold X_p "count" the genus g worldsheet instantons of degree n in X_p and arise when the fiber D-branes are ignored. The geometrical meaning of the generalized invariants $\mathrm{N}^g{}_{n,k}(p)$ for $k > 0$ will be elucidated later on.

Let us specialize at this point to the genus zero contribution, as it is this term which will exhibit the interesting phase structure of the theory. The higher genus contributions are similarly dealt with (The genus one free energy is worked out in appendix B). By parameterizing the free energy as

$$\mathcal{F}_0\big(t\,,\,\hat{t}\,;\,p\big) = \Big(1 - \mathrm{e}^{-2\,\hat{t}}\Big)^2 \sum_{n=1}^{\infty} \mathrm{e}^{-n\,t}\ F_n\big(\hat{t}\,;\,p\big)\ , \qquad (2.25)$$

we find that the first six contributions are given by

$$\begin{aligned}
F_1 &= 1\ , \\
F_2 &= \frac{\mathrm{e}^{-4\,\hat{t}}}{8}\,\Big[1 + 4\,p + 2\,p^2 + \mathrm{e}^{2\,\hat{t}}\,\big(2 - 4\,p^2\big) + \mathrm{e}^{4\,\hat{t}}\,\big(1 - 4\,p + 2\,p^2\big)\Big]\ , \\
F_3 &= \frac{\mathrm{e}^{-8\,\hat{t}}}{54}\,\Big[2 + 18\,p + 45\,p^2 + 36\,p^3 + 9\,p^4 + 6\ \mathrm{e}^{4\,\hat{t}}\,\big(1 - 9\,p^2 + 9\,p^4\big) \\
&\quad +\ \mathrm{e}^{8\,\hat{t}}\,\big(2 - 18\,p + 45\,p^2 - 36\,p^3 + 9\,p^4\big) + 2\ \mathrm{e}^{6\,\hat{t}}\,\big(2 - 9\,p - 9\,p^2 + 36\,p^3 - 18\,p^4\big) \\
&\quad + 2\ \mathrm{e}^{2\,\hat{t}}\,\big(2 + 9\,p - 9\,p^2 - 36\,p^3 - 18\,p^4\big)\Big]\ ,
\end{aligned}$$

$$
\begin{aligned}
F_4 = \frac{\mathrm{e}^{-12\,\hat{t}}}{192} \Big[& 3 + 44\,p + 214\,p^2 + 448\,p^3 + 432\,p^4 + 192\,p^5 + 32\,p^6 \\
& + \ \mathrm{e}^{10\,\hat{t}} \left(6 - 56\,p + 60\,p^2 + 416\,p^3 - 1008\,p^4 + 768\,p^5 - 192\,p^6\right) \\
& + \ \mathrm{e}^{12\,\hat{t}} \left(3 - 44\,p + 214\,p^2 - 448\,p^3 + 432\,p^4 - 192\,p^5 + 32\,p^6\right) \\
& + 2\ \ \mathrm{e}^{2\,\hat{t}} \left(3 + 28\,p + 30\,p^2 - 208\,p^3 - 504\,p^4 - 384\,p^5 - 96\,p^6\right) \\
& + 4\ \ \mathrm{e}^{6\,\hat{t}} \left(3 - 62\,p^2 + 216\,p^4 - 160\,p^6\right) \\
& + \ \mathrm{e}^{8\,\hat{t}} \left(9 - 44\,p - 150\,p^2 + 512\,p^3 + 144\,p^4 - 960\,p^5 + 480\,p^6\right) \\
& + \ \mathrm{e}^{4\,\hat{t}} \left(9 + 44\,p - 150\,p^2 - 512\,p^3 + 144\,p^4 + 960\,p^5 + 480\,p^6\right) \Big] \ ,
\end{aligned}
$$

$$
\begin{aligned}
F_5 = \frac{\mathrm{e}^{-16\,\hat{t}}}{3000} \Big[& 24 + 500\,p + 3750\,p^2 + 13500\,p^3 + 25875\,p^4 + 27500\,p^5 + 16250\,p^6 + 5000\,p^7 \\
& + 625\,p^8 + \ \mathrm{e}^{14\,\hat{t}} \big(48 - 700\,p + 2250\,p^2 + 5000\,p^3 - 38750\,p^4 + 76700\,p^5 - 69500\,p^6 \\
& + 30000\,p^7 - 5000\,p^8\big) + \ \mathrm{e}^{16\,\hat{t}} \big(24 - 500\,p + 3750\,p^2 - 13500\,p^3 + 25875\,p^4 \\
& - 27500\,p^5 + 16250\,p^6 - 5000\,p^7 + 625\,p^8\big) - 2\ \ \mathrm{e}^{2\,\hat{t}} \big(- 24 - 350\,p - 1125\,p^2 \\
& + 2500\,p^3 + 19375\,p^4 + 38350\,p^5 + 34750\,p^6 + 15000\,p^7 + 2500\,p^8\big) - 10\ \ \mathrm{e}^{8\,\hat{t}} \big(12 \\
& - 450\,p^2 + 3325\,p^4 - 7250\,p^6 + 4375\,p^8\big) + 4\ \ \mathrm{e}^{12\,\hat{t}} \big(18 - 175\,p - 125\,p^2 + 3500\,p^3 \\
& - 5125\,p^4 - 7950\,p^5 + 23000\,p^6 - 17500\,p^7 + 4375\,p^8\big) + 4\ \ \mathrm{e}^{4\,\hat{t}} \big(18 + 175\,p \\
& - 125\,p^2 - 3500\,p^3 - 5125\,p^4 + 7950\,p^5 + 23000\,p^6 + 17500\,p^7 + 4375\,p^8\big) \\
& + 2\ \ \mathrm{e}^{10\,\hat{t}} \big(48 - 250\,p - 1625\,p^2 + 5500\,p^3 + 8375\,p^4 - 28250\,p^5 - 1250\,p^6 \\
& + 35000\,p^7 - 17500\,p^8\big) + 2\ \ \mathrm{e}^{6\,\hat{t}} \big(48 + 250\,p - 1625\,p^2 - 5500\,p^3 \\
& + 8375\,p^4 + 28250\,p^5 - 1250\,p^6 - 35000\,p^7 - 17500\,p^8\big) \Big] \ ,
\end{aligned}
$$

$$
\begin{aligned}
F_6 = \frac{\mathrm{e}^{-20\,\hat{t}}}{2160} \Big[& 10 + 274\,p + 2837\,p^2 + 14940\,p^3 + 44955\,p^4 + 81756\,p^5 + 92250\,p^6 + 64800\,p^7 \\
& + 27540\,p^8 + 6480\,p^9 + 648\,p^{10} + \ \mathrm{e}^{18\,\hat{t}} \big(20 - 404\,p + 2230\,p^2 + 630\,p^3 - 44880\,p^4 \\
& + 170154\,p^5 - 304530\,p^6 + 303120\,p^7 - 171720\,p^8 + 51840\,p^9 - 6480\,p^{10}\big) \\
& + \ \mathrm{e}^{20\,\hat{t}} \big(10 - 274\,p + 2837\,p^2 - 14940\,p^3 + 44955\,p^4 - 81756\,p^5 + 92250\,p^6 - 64800\,p^7 \\
& + 27540\,p^8 - 6480\,p^9 + 648\,p^{10}\big) + 2\ \ \mathrm{e}^{2\,\hat{t}} \big(10 + 202\,p + 1115\,p^2 - 315\,p^3 \\
& - 22440\,p^4 - 85077\,p^5 - 152265\,p^6 - 151560\,p^7 - 85860\,p^8 - 25920\,p^9 + 3240\,p^{10}\big) \\
& + 3\ \ \mathrm{e}^{16\,\hat{t}} \big(10 - 148\,p + 225\,p^2 + 3780\,p^3 - 15035\,p^4 + 3168\,p^5 + 72610\,p^6 - 152640\,p^7 \\
& + 136620\,p^8 - 58320\,p^9 + 9720\,p^{10}\big) + 3\ \ \mathrm{e}^{4\,\hat{t}} \big(10 + 148\,p + 225\,p^2 - 3780\,p^3 - 15035\,p^4 \\
& - 3168\,p^5 + 72610\,p^6 + 152640\,p^7 + 136620\,p^8 + 58320\,p^9 + 9720\,p^{10}\big) \\
& + 12\ \ \mathrm{e}^{10\,\hat{t}} \big(5 - 297\,p^2 + 3760\,p^4 - 16325\,p^6 + 26460\,p^8 - 13608\,p^{10}\big) \\
& + 4\ \ \mathrm{e}^{14\,\hat{t}} \big(10 + 101\,p - 285\,p^2 + 3555\,p^3 - 1860\,p^4 - 26109\,p^5 + 42780\,p^6 \\
& + 24120\,p^7 - 100440\,p^8 - 77760\,p^9 - 19440\,p^{10}\big) + 4\ \ \mathrm{e}^{6\,\hat{t}} \big(10 + 101\,p - 285\,p^2 \\
& - 3555\,p^3 - 1860\,p^4 + 26109\,p^5 + 42780\,p^6 - 24120\,p^7 - 100440\,p^8 - 77760\,p^9
\end{aligned}
$$

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$$
\begin{aligned}
&- 19440\, p^{10}\big) + 2\ \mathrm{e}^{\,12\,\hat t}\big(25 - 137\, p - 1410\, p^2 + 4860\, p^3 + 14955\, p^4 - 45738\, p^5 \\
&- 39360\, p^6 + 146160\, p^7 - 11340\, p^8 - 136080\, p^9 + 68040\, p^{10}\big) + 2\ \mathrm{e}^{\,8\,\hat t}\big(25 \\
&+ 137\, p - 1410\, p^2 - 4860\, p^3 + 14955\, p^4 + 45738\, p^5 - 39360\, p^6 \\
&-146160\, p^7 - 11340\, p^8 + 136080\, p^9 + 68040\, p^{10}\big)\Big] \ . \qquad (2.26)
\end{aligned}
$$

Note that ${\mathrm{N}^0}_{n,k}(p)$ is a polynomial of degree $2n-2$ in p.

As a first application of these results, let us check the consistency of our computations with ordinary large N Yang-Mills theory. It is possible to find a limit in which the undeformed theory is recovered [34, 38, 39], namely $p \to \infty$ with $A = 2\, p\, \hat t = p\, N\, g_s$ kept fixed. The free energies (2.26) in this limit should reproduce the analogous quantities obtained in ordinary chiral Yang-Mills theory from the Gross-Taylor string expansion. By explicitly performing this limit we find

$$
\Phi_0(A) := \lim_{\substack{p\to\infty \\ A=2\,p\,\hat t}} \mathcal{F}_0\big(t\,,\,\hat t\,;\,p\big) = \sum_{n=1}^{\infty} \mathrm{e}^{\,-n\,A/2}\ \phi_n(A) \qquad (2.27)
$$

with

$$
\begin{aligned}
\phi_1 &= 1\ , \\
\phi_2 &= \frac12 - A + \frac14\, A^2\ , \\
\phi_3 &= \frac13 - 2\,A + 3\,A^2 - \frac43\, A^3 + \frac16\, A^4\ , \\
\phi_4 &= \frac14 - 3\,A + \frac{21}{2}\, A^2 - \frac{43}{3}\, A^3 + \frac{33}{4}\, A^4 - 2\,A^5 + \frac16\, A^6\ , \\
\phi_5 &= \frac15 - 4\,A + 25\,A^2 - \frac{202}{3}\, A^3 + \frac{529}{6}\, A^4 - \frac{883}{15}\, A^5 + \frac{121}{6}\, A^6 - \frac{10}{3}\, A^7 + \frac{5}{24}\, A^8\ , \\
\phi_6 &= \frac16 - 5\,A + \frac{195}{4}\, A^2 - \frac{647}{3}\, A^3 + \frac{1489}{3}\, A^4 - \frac{3178}{5}\, A^5 + \frac{1871}{4}\, A^6 - \frac{598}{3}\, A^7 \\
&\quad + 48\,A^8 - 6\,A^9 + \frac{3}{10}\, A^{10}\ . \qquad (2.28)
\end{aligned}
$$

We find exact agreement at this order with the results obtained in [40, 41] for the ordinary chiral QCD_2 string.

The second application is somewhat more sophisticated and will be discussed in sections 2.6 and 4. From (2.26) it is possible to compute the generalized Gromov-Witten invariants of the topological string theory on X_p. Because ordinary chiral QCD_2 computes Hurwitz numbers which are encoded in the coefficients of the area polynomials in (2.28) [40], our limiting procedure establishes a direct link between the combinatorics of branched covering maps to $\mathbb{P}^1$ and the geometry of rational curves embedded in Calabi-Yau threefolds. In section 4 we will use the information encoded in these free energies to estimate the radius of convergence of the string perturbation series and to describe the phase transitions in the chiral gauge theory.

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2.3 Toric geometry and the topological vertex

We will now derive the toric description of the local Calabi-Yau threefold (2.1), and by using the formalism of the topological vertex [11, 42] we will find the toric geometry associated to the large N limit of the chiral q-deformed Yang-Mills theory. The total space X_p may be regarded as a special lagrangian $\mathbb{T}^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$. With $(z_i)_{i=1}^4$ coordinates on the complex linear space $\mathbb{C}^4$, consider the real equation

$$\mu_p := |z_1|^2 + |z_4|^2 - p|z_2|^2 + (p-2)|z_3|^2 = t \tag{2.29}$$

and the U(1) group action on $\mathbb{C}^4$ given by

$$\big(z_1\,,\,z_4\,,\,z_2\,,\,z_3\big) \longmapsto \big(\mathrm{e}^{\,\mathrm{i}\,\alpha}\,z_1\,,\ \mathrm{e}^{\,\mathrm{i}\,\alpha}\,z_4\,,\ \mathrm{e}^{\,-\,\mathrm{i}\,p\,\alpha}\,z_2\,,\ \mathrm{e}^{\,\mathrm{i}\,(p-2)\alpha}\,z_3\big)\ . \tag{2.30}$$

Then $X_p = \mu_p^{-1}(t)/\,\mathrm{U}(1)$. For $z_2 = z_3 = 0$ eq. (2.29) describes a sphere whose area is proportional to t. Thus (z_1, z_4) can be taken as homogeneous coordinates for the base $\mathbb{P}^1$ of the fibration (2.1), while (z_2, z_3) may be regarded as coordinates of the fibers.

This realization defines X_p as a symplectic quotient. For this, we regard X_p as a union of local coordinate patches each symplectomorphic to $\mathbb{C}^3$ [42, 11]. There are two patches because the base $\mathbb{P}^1$ is given by the equation $|z_1|^2 + |z_4|^2 = t$ with one of z_1 or z_4 non-zero. In each patch we write down moment maps $(r_\alpha, r_\beta, r_\gamma)$ whose image gives global coordinates for the base $\mathbb{R}^3$ and which generate three hamiltonian flows on $\mathbb{C}^3$ with respect to its standard symplectic structure. The torus fiber $\mathbb{T}^2$ corresponds to the circle actions generated by r_α and r_β, while r_γ generates the real line $\mathbb{R}$. The local $\mathbb{C}^3$ geometry of X_p is represented by an oriented trivalent planar graph which encodes the degeneration locus of the fibration in the base $\mathbb{R}^3$, drawn in the $r_\gamma = 0$ plane. An edge of the graph is labelled by an integer vector $(n, m) \in \mathbb{Z}^2$ that corresponds to the generator of the homology group $H_1(\mathbb{T}^2, \mathbb{Z})$ which is the shrinking cycle. Changing the edge orientations sends $(n, m) \mapsto (-n, -m)$ and does not alter the Calabi-Yau geometry. The local geometry is described by assigning to each patch three integer vectors $\vec{v}_a = (n_a, m_a)$, $a = 1, 2, 3$ which single out the degenerating cycles unambiguously up to $\mathrm{SL}(2, \mathbb{Z})$ modular transformations of $\mathbb{T}^2$. The lines in the base $\mathbb{R}^3$ where the $\mathbb{T}^2$ fibers degenerate are correlated with the zeroes of the corresponding moment maps. The Calabi-Yau condition is encoded in the requirement $\sum_a \vec{v}_a = (0, 0)$. The conditions $|\vec{v}_a \wedge \vec{v}_b| = 1$, $a < b$ ensure smoothness of X_p, where $\wedge$ denotes the symplectic product on the vector space $H_1(\mathbb{T}^2, \mathbb{R})$.

Let us now explicitly display the two $\mathbb{C}^3$ patches of the Calabi-Yau space X_p.

$z_4 \neq 0$: In this case we can use (2.29) to solve for the modulus of z_4 in terms of z_1, z_2, z_3 and gauge away its phase by dividing by the U(1) action (2.30) of the symplectic quotient construction. This defines the patch $U_4(z_1, z_2, z_3) \cong \mathbb{C}^3$. The hamiltonians which generate the homology cycles of the $\mathbb{T}^2$ fiber are defined by

$$\begin{aligned} r_\alpha &= |z_2|^2 - |z_1|^2\ , \\ r_\beta &= |z_3|^2 - |z_1|^2 \end{aligned} \tag{2.31}$$

with the torus action

$$\mathrm{e}^{\,\mathrm{i}\,\alpha\, r_\alpha+\mathrm{i}\,\beta\, r_\beta} \,:\, \big(z_1\,,\,z_2\,,\,z_3\big) \,\longmapsto\, \big(\mathrm{e}^{\,-\,\mathrm{i}\,(\alpha+\beta)}\,z_1\,,\ \mathrm{e}^{\,\mathrm{i}\,\alpha}\,z_2\,,\ \mathrm{e}^{\,\mathrm{i}\,\beta}\,z_3\big)\ . \tag{2.32}$$

The $\mathbb{R}$ fiber is generated by $r_\gamma = \mathrm{Im}(z_1z_2z_3z_4)$. The degeneration locus corresponds to zero sets of these moment maps. Using (2.29) one finds that the $(1,0)$ cycle generated by r_β degenerates over the line $z_1 = z_3 = 0$ where $r_\beta = r_\gamma = 0$ and $r_\alpha \geq 0$, the $(0,1)$ cycle generated by r_α degenerates over $z_1 = z_2 = 0$ where $r_\alpha = r_\gamma = 0$ and $0 \leq r_\beta < t$, and the $(-1,-1)$ cycle generated by $-(r_\alpha + r_\beta)$ degenerates over $z_2 = z_3 = 0$ where $r_\alpha - r_\beta = r_\gamma = 0$ and $-t < r_\alpha \leq 0$. Imposing the Calabi-Yau and smoothness conditions, and defining $\vec{v}_1 = (-1,-1)$, we arrive at the basis for the degeneration locus in $H_1(\mathbb{T}^2,\mathbb{Z})$ given by

$$\vec{v}_1 = (-1,-1)\ ,\qquad \vec{v}_2 = (0,1)\ ,\qquad \vec{v}_3 = (1,0)\ . \tag{2.33}$$

The graph representing this locus is depicted in figure 1.

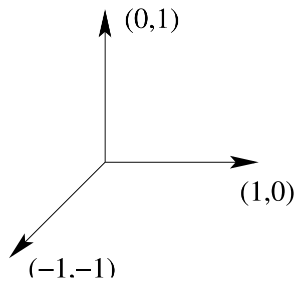

Figure 1: Toric graph for the patch $U_4(z_1,z_2,z_3)$ of X_p representing the singular locus in the base $\mathbb{R}^3$ with global coordinates $(r_\alpha, r_\beta, r_\gamma)$.

$\boldsymbol{z_1 \neq 0}$**:** In this case we can solve for z_1 and produce the patch $U_1(z_2,z_3,z_4) \cong \mathbb{C}^3$. The same hamiltonians as before generate the $\mathbb{T}^2 \times \mathbb{R}$ fiber, except that now z_1 is no longer a natural coordinate for the patch and so we use (2.29) to write

$$\begin{aligned} r_\alpha &= |z_2|^2 - |z_1|^2 = -t + (1-p)|z_2|^2 + (p-2)|z_3|^2 + |z_4|^2\ ,\\ r_\beta &= |z_3|^2 - |z_1|^2 = -t - p|z_2|^2 + (p-1)|z_3|^2 + |z_4|^2 \end{aligned} \tag{2.34}$$

with the torus action

$$\mathrm{e}^{\,\mathrm{i}\,\alpha\, r_\alpha+\mathrm{i}\,\beta\, r_\beta} \,:\, \big(z_2\,,\,z_3\,,\,z_4\big) \,\longmapsto\, \big(\mathrm{e}^{\,\mathrm{i}\,(1-p)\alpha-\,\mathrm{i}\,p\,\beta}\,z_2\,,\ \mathrm{e}^{\,\mathrm{i}\,(p-2)\alpha+\,\mathrm{i}\,(p-1)\beta}\,z_3\,,\ \mathrm{e}^{\,\mathrm{i}\,(\alpha+\beta)}\,z_4\big)\ . \tag{2.35}$$

Again by using (2.29) one finds that the $(p-2,p-1)$ cycle generated by $(p-2)r_\alpha+(p-1)r_\beta$ degenerates over the line $z_2 = z_4 = 0$ where $(p-1)r_\alpha + (p-2)r_\beta = (3-2p)t$, $r_\gamma = 0$ and $-t \leq r_\alpha < 0$, the $(1-p,-p)$ cycle degenerates for $z_3 = z_4 = 0$ where $p\,r_\alpha + (p-1)r_\beta = (1-2p)t$, $r_\gamma = 0$ and $r_\alpha \leq t$, and finally the $(1,1)$ cycle generated by $r_\alpha + r_\beta$ degenerates over $z_2 = z_3 = 0$ where $r_\alpha - r_\beta = r_\gamma = 0$ and $-t \le r_\alpha < 0$. Defining $\vec{v}_1^{\,\prime} = (1,1)$ thereby gives the degeneration basis

$$\vec{v}_1^{\,\prime} = (1,1) \ , \qquad \vec{v}_2^{\,\prime} = (1-p,-p) \ , \qquad \vec{v}_3^{\,\prime} = (p-2,p-1) \tag{2.36}$$

depicted in figure 2.

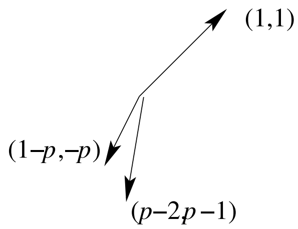

Figure 2: Toric graph for the patch $U_1(z_2,z_3,z_4)$ of X_p representing the singular locus in the base $\mathbb{R}^3$ with global coordinates $(r_\alpha, r_\beta, r_\gamma)$.

Note that both patches share the common edge where $z_2 = z_3 = 0$ through the orientation reversing symmetry $(-1,-1) \leftrightarrow (1,1)$ of their graphs. The length of this edge is the Kähler parameter t and it represents the base $\mathbb{P}^1$ of the fibration (2.1). The threefold X_p is finally obtained by gluing the two $\mathbb{C}^3$ patches together along this common edge. The transition functions are given by $\mathrm{SL}(2,\mathbb{Z})$ modular transformations of the $\mathbb{T}^2$ fibers between the patches. The graph encoding the toric geometry of X_p is depicted in figure 3.

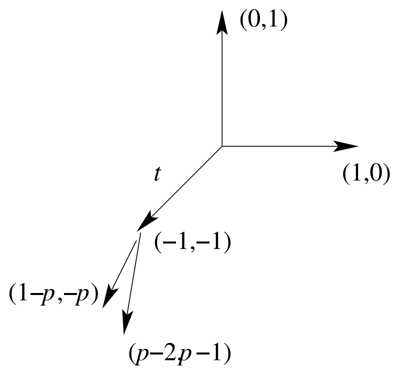

Figure 3: Toric diagram of $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \longrightarrow \mathbb{P}^1$. The manifold is built by gluing its two $\mathbb{C}^3$ patches together along their common but oppositely oriented sphere $\mathbb{P}^1$ with Kähler modulus t.

We can now construct generic topological string amplitudes on X_p by using the formalism of the topological vertex [42] (see [11] for a review). The toric geometry of X_p is encoded in a planar graph obtained by gluing trivalent vertices representing the $\mathbb{C}^3$ patches. The basic object associated to the trivalent vertices is the open topological string vacuum amplitude $C_{\hat{R}_{(1)}\hat{R}_{(2)}\hat{R}_{(3)}}(q)$ on the trivial $\mathbb{C}^3$ geometry, where $\hat{R}_{(a)}$, $a = 1,2,3$ are $\mathrm{SU}(N)$ representation labels on the three edges $\vec{v}_a$ of the graph. This defines the cubic topological vertex [42]. It is proportional to the combinatorial quantity $\sum_{\mathcal{Y}} q^{|\mathcal{Y}|}$, where the sum runs over plane partitions $\mathcal{Y}$ whose edges in the three directions correspond to Young tableaux with the shapes $\hat{R}_{(a)}$ [45]. Explicitly, the topological vertex amplitude in the canonical framing (2.33) is given by

$$C_{\hat{R}_{(1)}\hat{R}_{(2)}\hat{R}_{(3)}}(q) = q^{(\kappa_{\hat{R}_{(2)}}+\kappa_{\hat{R}_{(3)}})/2} \sum_{\hat{Q}} \sum_{\hat{Q}_{(1)},\hat{Q}_{(3)}} N_{\hat{Q}\hat{Q}_{(1)}}{}^{\hat{R}_{(1)}} N_{\hat{Q}\hat{Q}_{(3)}}{}^{\hat{R}_{(1)}^\top} \times \frac{W_{\hat{R}_{(2)}^\top\hat{Q}_{(1)}}(q)\, W_{\hat{R}_{(2)}\hat{Q}_{(3)}}(q)}{W_{\hat{R}_{(2)}}(q)} \tag{2.37}$$

where $N_{\hat{Q}\hat{R}}{}^{\hat{S}}$ are the SU(N) fusion numbers. It is natural to expect that closed string amplitudes associated to the toric diagram in figure 3 can be computed by gluing the open topological string amplitudes associated to the trivalent vertices, in much the same way that one computes amplitudes in perturbative quantum field theory by gluing vertices through propagators.

The gluing rules for the topological vertex are quite simple [42]. First of all, we need to reverse the orientation of the edge $\vec{v}_1$, which in the open topological string amplitude induces the transformation

$$C_{\hat{R}_{(1)}\hat{R}_{(2)}\hat{R}_{(3)}}(q) \longmapsto (-1)^{|\hat{R}_{(1)}|}\, C_{\hat{R}_{(1)}^\top\hat{R}_{(2)}\hat{R}_{(3)}}(q) \tag{2.38}$$

corresponding to the gluing of topological branes to antibranes. Then we have to take care of the fact that the patch U_1 is not in the canonical framing given by the basis (2.33) [42, 11]. This implies the presence of an additional factor $(-1)^{-n_1|\hat{R}_{(1)}|}\, q^{n_1\kappa_{\hat{R}_{(1)}}/2}$ in the amplitude, where $n_1 = |\vec{v}_3' \wedge \vec{v}_3| = p-1$. We glue together the two vertices with the Schwinger propagator $\mathrm{e}^{-|\hat{R}_{(1)}|t}\, \delta_{\hat{R}_{(1)}',\hat{R}_{(1)}^\top}$ coming from the worldsheet instanton action on the lines which represent spheres $\mathbb{P}^1$. Collecting all factors, we arrive at the topological string partition function given by

$$Z_{\substack{\hat{R}_{(2)},\hat{R}_{(3)}\\ \hat{R}_{(2)}',\hat{R}_{(3)}'}}(t;p) = \sum_{\hat{R}_{(1)}} \mathrm{e}^{-|\hat{R}_{(1)}|t}\, (-1)^{p|\hat{R}_{(1)}|}\, q^{(p-1)\kappa_{\hat{R}_{(1)}}/2}\, C_{\hat{R}_{(1)}\hat{R}_{(2)}\hat{R}_{(3)}}(q)\, C_{\hat{R}_{(1)}^\top\hat{R}_{(2)}'\hat{R}_{(3)}'}(q)\ . \tag{2.39}$$

This is a generalization of the closed topological string vacuum amplitude on X_p, with representations $\hat{R}_{(2)}, \hat{R}_{(2)}', \hat{R}_{(3)}, \hat{R}_{(3)}'$ placed on the external legs of the toric diagram. These representations describe D-brane degrees of freedom [42] corresponding to non-compact special lagrangian submanifolds with the topology of $\mathbb{C}\times S^1$ in the edges that go to infinity in the toric diagram for X_p.

Let us now analytically continue $t = \mathrm{i}\,\pi\, p + t'$. As explained in [28, 43] there are only two stacks of D-branes inserted in the fibers of X_p that correspond to extra closed string moduli coming from infinity. In the other two directions, where the D4-branes are wrapped, we have to consider trivial representations (heuristically this line bundle should be understood as a degenerate limit of a compact cycle). These are the cycles $\vec{v}_3$ and $\vec{v}_2'$ and the condition requires setting $\hat{R}_3 = \hat{R}_2' = 0$. By using the identity

$$W_{\hat{R}\hat{S}^\top}(q) = q^{-\kappa_{\hat{S}}/2}\, C_{0\hat{R}\hat{S}}(q) \tag{2.40}$$

along with the cyclicity of the topological vertex in its representation labels, we arrive at

$$Z_{\substack{\hat{R}_{(2)},\,0\\ 0\,,\hat{R}'_{(3)}}}(t';p) = q^{\kappa_{\hat{R}_{(2)}}/2} \sum_{\hat{R}_{(1)}} \mathrm{e}^{\,-|\hat{R}_{(1)}|t'}\, q^{(p-2)\kappa_{\hat{R}_{(1)}}/2}\, W_{\hat{R}'_{(3)}\hat{R}_{(1)}}(q)\, W_{\hat{R}_{(1)}\hat{R}^{\top}_{(2)}}(q)\ . \tag{2.41}$$

Up to overall normalization this expression coincides with the chiral partition function $\mathcal{Z}^{q\mathrm{YM},+}_{\hat{R}'_{(3)},\hat{R}_{(2)}}(t'\,)$ computed in [28] (see eq. (2.8)). In particular, the closed topological string partition function on X_p is given by

$$Z_{\substack{0,0\\0,0}}(t';p) = \sum_{\hat{R}_{(1)}} \mathrm{e}^{\,-|\hat{R}_{(1)}|t'}\, q^{(p-2)\kappa_{\hat{R}_{(1)}}/2}\, W_{\hat{R}_{(1)}}(q)^2\ . \tag{2.42}$$

This expression coincides exactly with the perturbative part of the topological string amplitude on X_p at the value of the Kähler parameter fixed by the attractor mechanism [28]. It is straightforward to check that the $k=0$ sector of eq. (2.22) also reproduces the topological string partition function on X_p. The relevant Gromov-Witten invariants are given in this case by $\mathrm{N}^g{}_n(p) = \mathrm{N}^g{}_{n,0}(p)$. We will now explore this relationship in more detail.

2.4 Chiral partition function as a topological string amplitude

In section 2.2 we found that the chiral partition function, obtained by disregarding coupled representations, has a free energy which is nicely organized as a double series expansion in the parameters t and $\hat{t}$ according to eq. (2.24). This strongly suggests that $\hat{t}$ should be regarded as another Kähler parameter. We will now show that this is indeed the case and also make contact with the amplitude (2.42) which can be interpreted as the topological string partition function on X_p. For this, let us begin with the simple identity

$$\prod_{i=1}^{i_{\max}}\ \prod_{j=1}^{n_i}\left(1 - q^{j-i}\ \mathrm{e}^{\,-g_sN}\right) = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\, f_{\hat{R}}(q^n)\ \ \mathrm{e}^{\,-n\,g_sN}\right] \tag{2.43}$$

where

$$f_{\hat{R}}(q) = \sum_{i=1}^{i_{\max}}\ \sum_{j=1}^{n_i} q^{j-i}\ . \tag{2.44}$$

This can in turn be written in terms of the function

$$N_{\hat{R}}\big(q\,,\ \mathrm{e}^{\,-g_sN}\big) := \sum_{\hat{S}} (-1)^{|\hat{S}|}\ \ \mathrm{e}^{\,-|\hat{S}|g_sN}\, W_{\hat{S}\hat{R}}(q)\, W_{\hat{S}^{\top}}(q) \tag{2.45}$$

by means of the identity [28]

$$N_{\hat{R}}\big(q\,,\ \mathrm{e}^{\,-g_sN}\big) = N_0(q,N)\, W_{\hat{R}}(q)\, \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\, f_{\hat{R}}(q^n)\ \ \mathrm{e}^{\,-n\,g_sN}\right]\ . \tag{2.46}$$

Along with (2.40) these identities enable us to rewrite our chiral partition function (2.20) as

$$\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}} = \mathsf{Z}_0(q) \sum_{\hat{R}}\ \sum_{\hat{R}_{(1)},\hat{R}_{(2)}} \mathrm{e}^{\,-t|\hat{R}|-t_1|\hat{R}_{(1)}|-t_2|\hat{R}_{(2)}|}\,(-1)^{|\hat{R}_{(1)}|+|\hat{R}_{(2)}|+p|\hat{R}|}\, q^{(p-1)\kappa_{\hat{R}}/2}$$

$$\times\, C_{0\hat{R}_{(1)}\hat{R}^\top}(q)\, C_{0\hat{R}\hat{R}_{(2)}}(q)\, C_{00\hat{R}^\top_{(1)}}(q)\, C_{00\hat{R}^\top_{(2)}}(q)\ , \tag{2.47}$$

where $t_1 = t_2 = 2\,\hat{t}$. In this computation we have restored the normalization factor S_{00} and defined (see appendix A)

$$\mathsf{Z}_0(q) = \lim_{N\to\infty} \frac{S_{00}(q,N)}{N_0(q,N)} = M(q)\,\eta(q)^N\, q^{-N/24}\ , \tag{2.48}$$

where $M(q)$ is the McMahon function and $\eta(q)$ is the Dedekind function. The factor (2.48) accounts for the contribution of constant string maps into X_p (having winding number $n=0$) and for quadratic (in t) ambiguities in the genus zero partition function [28], plus some non-perturbative corrections. The remaining part of (2.47) represents instead the full perturbative contribution to a closed topological string amplitude.

The relevant toric diagram can be read off directly from eq. (2.47) by reversing the topological vertex gluing rules. It contains four vertices connected by three framed edges with Kähler parameters t_1, t and t_2 (figure 4). Recall from section 2.3 that the $\vec{v}_2$ and $\vec{v}_3'$

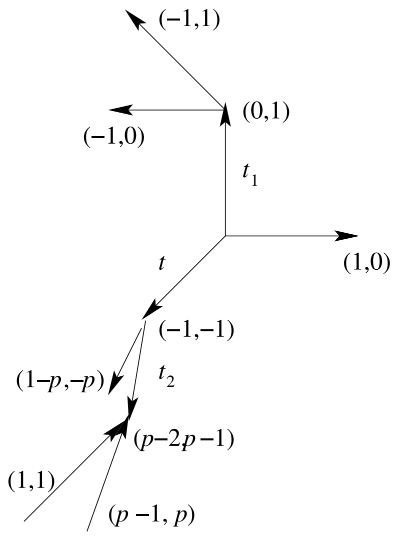

Figure 4: Toric diagram describing the topological string expansion of the chiral q-deformed gauge theory partition function $\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}}$.

edges of this toric graph represent one-cycles in the fibers of the toric geometry of X_p and have length t. Gluing the open topological string vertex to these edges thus corresponds to the insertion of D-branes in the fiber of (2.1). In this local $\mathbb{C}^3$ patch these are special lagrangian submanifolds with the topology of $\mathbb{C}\times S^1$ corresponding to D-branes wrapping an S^1 cycle in the fiber. The gluing edges labelled by t_a, $a=1,2$ thus correspond to rational holomorphic curves $\Sigma_a \cong \mathbb{P}^1$ at distances t_a from the sphere $\mathbb{P}^1$ corresponding to the gluing edge labelled t which represents the base of the fibration (2.1). The t_a themselves are the Kähler parameters of the corresponding curves Σ_a.

This defines a nonsingular toric Calabi-Yau scheme $\hat{X}_p$. It can be naturally thought of as emerging from a large N geometric transition from the toric threefold described by a nonplanar trivalent graph with four vertices [46]–[48]. This geometry locally contains two three-sphere cotangent bundles T^*S^3 each constructed as a $\mathbb{T}^2$ fibration over an interval and glued together along a $\mathbb{P}^1$, and it corresponds to a surgery construction on S^3 wherein one performs a Heegaard split of S^3 along solid tori and glues the tori together along their boundary $\mathbb{T}^2$ through the $\mathrm{SL}(2,\mathbb{Z})$ transformation that relates one of the collapsing cycles of the toric geometry to the other. Since this geometry is not globally the cotangent bundle of a three-manifold, the topological string dynamics is described by two $\mathrm{U}(N)$ Chern-Simons theories on S^3, along with an additional sector of open strings stretched between the two three-spheres which correspond to non-degenerate holomorphic instantons. The geometric transition consists of shrinking the two three-spheres to points, and then resolving these singularities with two copies of $\mathbb{P}^1$ to realize the toric manifold $\hat{X}_p$ depicted in figure 4. This description exhibits the two stacks of D-branes present in the geometry explicitly (seen here as wrapping the lagrangian submanifolds $S^3 \subset T^*S^3$ before the transition), and it may also account for the generic discrepancy between the Chern-Simons and q-deformed Yang-Mills descriptions of topological strings on X_p [34].

It is instructive to compare this result with the black hole partition function in the sector of vanishing Ramond-Ramond flux. From the large N expansion of the q-deformed partition function for the full coupled theory one finds from (2.5), (2.6) and (2.8) the result [28]

$$
\begin{aligned}
Z^0_{\mathrm{BH}} = \big|\mathsf{Z}_0(q)\big|^2 \sum_{\hat{R}_\pm} & \mathrm{e}^{\,-t|\hat{R}_+|-\bar{t}\,|\hat{R}_-|}\,(-1)^{p|\hat{R}_+|+p|\hat{R}_-|}\,q^{(p-1)(\kappa_{\hat{R}_+}+\kappa_{\hat{R}_-})/2} \\
&\times \sum_{\hat{R}_{(1)},\hat{R}_{(2)}} \mathrm{e}^{\,-2\,\mathrm{Re}(t_1)|\hat{R}_{(1)}|}\;\mathrm{e}^{\,-\mathrm{Re}(t_2)|\hat{R}_{(2)}|}\,(-1)^{|\hat{R}_{(1)}|+|\hat{R}_{(2)}|} \\
&\times C_{0\hat{R}_{(1)}\hat{R}_+^\top}(q)\,C_{0\hat{R}_+\hat{R}_{(2)}}(q)\,C_{0\hat{R}_-\hat{R}_{(1)}^\top}(q)\,C_{0\hat{R}_-^\top\hat{R}_{(2)}^\top}(q)\ .
\end{aligned}
\tag{2.49}
$$

We see that the chiral partition function (2.47) corresponds to the contribution to (2.49) from the trivial sector $\hat{R}_- = 0$. This is natural in the toric description, as it corresponds to dropping a gluing edge in the construction of the corresponding threefold. Note that our Kähler modulus t is real as we have not included a θ-angle in the definition of the original q-deformed gauge theory.

2.5 Gromov-Witten invariants

We will now analyse the Gromov-Witten invariants of the threefold $\hat{X}_p$. Modulo torsion this space has, by construction, second homology group $H_2(\hat{X}_p,\mathbb{Z}) \cong \mathbb{Z}^3$ with generating two-cycles carrying the Kähler parameters $\boldsymbol{t} := (t_1, t, t_2)$. Holomorphic string maps, which generate the chiral theory, preserve orientations of curves and are classified by *effective* degrees taking values in $H_2(\hat{X}_p,\mathbb{Z})_+ \cong \mathbb{N}_0^3$. The structure of eq. (2.47) suggests that the chiral expansion of the free energy $\mathcal{F}^{q\mathrm{YM}}_{\mathrm{chiral}}$ should be understood as a particular contribution to the more general free energy

$$
\mathcal{F}_{\hat{X}_p} = \sum_{g=0}^\infty g_s^{2g-2}\ \hat{\mathcal{F}}_g \tag{2.50}
$$

for topological A-model strings on $\hat{X}_p$. The free energy at genus g is given by a sum over effective two-homology classes of genus g worldsheet instantons as

$$\hat{\mathcal{F}}_g(\boldsymbol{t};p) = \sum_{\boldsymbol{n}\in H_2(\hat{X}_p,\mathbb{Z})_+} \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{t}}\, \hat{\mathrm{N}}^g{}_{\boldsymbol{n}}(p) \ , \tag{2.51}$$

where $\hat{\mathrm{N}}^g{}_{\boldsymbol{n}}(p) \in \mathbb{Q}$ are the Gromov-Witten invariants of $\hat{X}_p$. The chiral amplitude is obtained by taking $t_1 = t_2 = 2\,\hat{t}$ and restricting the sums (2.51) to the sector where the effective degree $\boldsymbol{n} := (k_1, n, k_2)$ is constrained by $k_1 + k_2 = k$ with $0 \leq k \leq 2n$. The expansion of the free energy (2.51) in the three independent Kähler parameters can be used to easily derive $\hat{\mathrm{N}}^g{}_{\boldsymbol{n}}(p)$ and the generalized Gromov-Witten invariants $\mathrm{N}^g{}_{n,k}(p)$ of X_p appearing in the large N limit of chiral q-deformed Yang-Mills theory through

$$\sum_{k_1=0}^{k} \hat{\mathrm{N}}^g{}_{(k_1,n,k-k_1)}(p) = \mathrm{N}^g{}_{n,k}(p) \ . \tag{2.52}$$

In particular, $\hat{\mathrm{N}}^g{}_{(0,n,0)}(p) = \mathrm{N}^g{}_n(p)$ are the Gromov-Witten invariants of the original threefold X_p. We will exhibit the first few invariants only for the genus $g = 0$ case. Higher genera are similarly dealt with.

The $n = 1$ terms in eq. (2.24) are of the form $\sum_{k=0}^{2} \mathrm{e}^{-t-2\,k\,\hat{t}}\ \mathrm{N}^g{}_{1,k}$. The required two-homology classes in (2.51) are given by

$$\boldsymbol{n} \ = \ (0,1,0) \ , \quad (1,1,0) \ , \quad (0,1,1) \ , \quad (1,1,1) \ , \tag{2.53}$$

and the corresponding Gromov-Witten invariants are

$$\hat{\mathrm{N}}^0{}_{(0,1,0)} = -\hat{\mathrm{N}}^0{}_{(1,1,0)} = -\hat{\mathrm{N}}^0{}_{(0,1,1)} = \hat{\mathrm{N}}^0{}_{(1,1,1)} = (-1)^p \ . \tag{2.54}$$

For $n = 2$, by expanding the sum explicitly it is again easy to write down the contributing degrees $\boldsymbol{n} \in H_2(\hat{X}_p, \mathbb{Z})_+$ and thus find the Gromov-Witten invariants

$$\begin{aligned}
\hat{\mathrm{N}}^0{}_{(0,2,0)} \ &= \ \frac{1}{8}\left(1 - 4\,p + 2\,p^2\right) \ , \\
\hat{\mathrm{N}}^0{}_{(1,2,0)} = \hat{\mathrm{N}}^0{}_{(0,2,1)} \ &= \ \frac{1}{2}\left(p - p^2\right) \ , \\
\hat{\mathrm{N}}^0{}_{(0,2,2)} = \hat{\mathrm{N}}^0{}_{(2,2,0)} \ &= \ -\frac{1}{8}\left(1 - 2\,p^2\right) \ , \\
\hat{\mathrm{N}}^0{}_{(1,2,1)} \ &= \ p^2 \ , \\
\hat{\mathrm{N}}^0{}_{(2,2,1)} = \hat{\mathrm{N}}^0{}_{(1,2,2)} \ &= \ -\frac{1}{2}\left(p + p^2\right) \ , \\
\hat{\mathrm{N}}^0{}_{(2,2,2)} \ &= \ \frac{1}{8}\left(1 + 4\,p + 2\,p^2\right) \ .
\end{aligned} \tag{2.55}$$

Finally, by following the same route for $n = 3$ we find the invariants

$$\hat{\mathrm{N}}^0{}_{(0,3,0)} \ = \ (-1)^p \left(\frac{1}{27} - \frac{1}{3}\,p + \frac{5}{6}\,p^2 - \frac{2}{3}\,p^3 + \frac{1}{6}\,p^4\right) \ ,$$

$$
\begin{aligned}
\hat{\mathrm{N}}^0{}_{(1,3,0)} = \hat{\mathrm{N}}^0{}_{(0,3,1)} &= (-1)^p \left(\frac{1}{6}\,p - p^2 + \frac{4}{3}\,p^3 - \frac{1}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(2,3,0)} = \hat{\mathrm{N}}^0{}_{(0,3,2)} &= (-1)^p \left(\frac{1}{6}\,p - \frac{2}{3}\,p^3 + \frac{1}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(1,3,1)} &= (-1)^p \left(\frac{1}{2}\,p^2 - 2\,p^3 + \frac{3}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(3,3,0)} = \hat{\mathrm{N}}^0{}_{(0,3,3)} &= -(-1)^p \left(\frac{1}{27} - \frac{1}{6}\,p^2 + \frac{1}{6}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(1,3,2)} &= (-1)^p \left(\frac{1}{2}\,p^2 - \frac{3}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(3,3,1)} = \hat{\mathrm{N}}^0{}_{(1,3,3)} &= -(-1)^p \left(\frac{1}{6}\,p - \frac{2}{3}\,p^3 - \frac{1}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(2,3,2)} &= (-1)^p \left(\frac{1}{2}\,p^2 + 2\,p^3 + \frac{3}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(3,3,2)} = \hat{\mathrm{N}}^0{}_{(2,3,3)} &= -(-1)^p \left(\frac{1}{6}\,p + p^2 + \frac{4}{3}\,p^3 + \frac{1}{2}\,p^4\right) \ , \\
\hat{\mathrm{N}}^0{}_{(3,3,3)} &= (-1)^p \left(\frac{1}{27} + \frac{1}{3}\,p + \frac{5}{6}\,p^2 + \frac{2}{3}\,p^3 + \frac{1}{6}\,p^4\right) \ .
\end{aligned} \tag{2.56}
$$

Let us now explore the hidden integrality structure of the Gromov-Witten invariants through the embedding of the topological A-model string theory into Type IIA string theory on $\hat{X}_p$ [49, 50]. The generating function (2.50) for the all-genus topological string amplitudes can be written as a generalized index that counts BPS states of D2-branes wrapping holomorphic curves in $\hat{X}_p$ as

$$
\mathcal{F}_{\hat{X}_p} = \sum_{g=0}^\infty \ \sum_{\boldsymbol{n}\in H_2(\hat{X}_p,\mathbb{Z})_+} \hat{\mathrm{n}}^g{}_{\boldsymbol{n}}(p) \sum_{d=1}^\infty \frac{1}{d} \left(2\,\sin\big(\frac{d\,g_s}{2}\big)\right)^{2g-2} \ \mathrm{e}^{-d\,\boldsymbol{n}\cdot\boldsymbol{t}} \ , \tag{2.57}
$$

where $\hat{\mathrm{n}}^g{}_{\boldsymbol{n}}(p) \in \mathbb{Z}$ are the Gopakumar-Vafa integer invariants which compute the Euler characteristic of the moduli space of embedded curves of genus g and two-homology class $\boldsymbol{n}$ in $\hat{X}_p$. As a consistency test of our interpretation of the chiral gauge theory as a topological string theory, we will now extract the invariants $\hat{\mathrm{n}}^0{}_{\boldsymbol{n}}(p)$ directly from the expansion of the genus zero free energy and verify that they are indeed integers. The case of higher genera can be similarly handled by using the explicit inversion formula between the two expansions (2.50), (2.51) and (2.57) that expresses the Gopakumar-Vafa invariants $\hat{\mathrm{n}}^g{}_{\boldsymbol{n}}$ in terms of Gromov-Witten invariants $\hat{\mathrm{N}}^h{}_{\boldsymbol{m}}$.

At genus zero we find by comparing (2.50), (2.51) and (2.57) the relation

$$
\hat{\mathrm{N}}^0{}_{\boldsymbol{n}}(p) = \sum_{d\,|\,\boldsymbol{n}} \frac{1}{d^3}\ \hat{\mathrm{n}}^0{}_{\boldsymbol{n}/d}(p) \ . \tag{2.58}
$$

We can invert this expression for a given effective class $\boldsymbol{n} \in H_2(\hat{X}_p,\mathbb{Z})_+$ to get the invariants $\hat{\mathrm{n}}^0{}_{\boldsymbol{n}}$. The first few identifications are simple. For example, the degree $\boldsymbol{n} = (1,1,0)$ has no divisors other than $d = 1$, and thus $\hat{\mathrm{n}}^0{}_{(1,1,0)}(p) = \hat{\mathrm{N}}^0{}_{(1,1,0)}(p) = -(-1)^p$. Instead, for the degree $\boldsymbol{n} = (2,2,0)$ we have

$$\hat{\mathrm{N}}^0{}_{(2,2,0)}(p) = \hat{\mathrm{n}}^0{}_{(2,2,0)}(p) + \frac{1}{2^3}\,\hat{\mathrm{n}}^0{}_{(1,1,0)}(p) \tag{2.59}$$

leading to

$$\hat{\mathrm{n}}^0{}_{(2,2,0)}(p) = -\frac{1}{8}\left(1 - 2\,p^2\right) + \frac{1}{2^3}\,(-1)^p$$

which is indeed an integer. Proceeding iteratively along these same lines, it is straightforward to derive the first few sets of Gopakumar-Vafa invariants. Omitting those that are obtained by the obvious symmetries, we obtain the list

$$\hat{\mathrm{n}}^0{}_{(0,1,0)} = (-1)^p = -\hat{\mathrm{n}}^0{}_{(1,1,0)} = \hat{\mathrm{n}}^0{}_{(1,1,1)}\ , \tag{2.60}$$

$$\begin{aligned}
\hat{\mathrm{n}}^0{}_{(0,2,0)} &= \frac{1}{8}\left(1 - (-1)^p - 4\,p + 2\,p^2\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(0,2,1)} &= \frac{1}{2}\left(p - p^2\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(2,2,0)} &= -\frac{1}{8}\left(1 - (-1)^p - 2\,p^2\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(1,2,1)} &= p^2\ ,\\
\hat{\mathrm{n}}^0{}_{(2,2,2)} &= \frac{1}{8}\left(1 - (-1)^p + 4\,p + 2\,p^2\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(2,2,1)} &= -\frac{1}{2}\left(p + p^2\right)\ ,
\end{aligned} \tag{2.61}$$

$$\begin{aligned}
\hat{\mathrm{n}}^0{}_{(0,3,0)} &= -(-1)^p\left(\frac{1}{3}\,p - \frac{5}{6}\,p^2 + \frac{2}{3}\,p^3 - \frac{1}{6}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(1,3,0)} &= (-1)^p\left(\frac{1}{6}\,p - p^2 + \frac{4}{3}\,p^3 - \frac{1}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(1,3,1)} &= (-1)^p\left(\frac{1}{2}\,p^2 - 2\,p^3 + \frac{3}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(2,3,0)} &= (-1)^p\left(\frac{1}{6}\,p - \frac{2}{3}\,p^3 + \frac{1}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(1,3,2)} &= (-1)^p\left(\frac{1}{2}\,p^2 - \frac{3}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(3,3,0)} &= (-1)^p\left(\frac{1}{6}\,p^2 - \frac{1}{6}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(2,3,2)} &= (-1)^p\left(\frac{1}{2}\,p^2 + 2\,p^3 + \frac{3}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(3,3,1)} &= -(-1)^p\left(\frac{1}{6}\,p - \frac{2}{3}\,p^3 - \frac{1}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(3,3,2)} &= -(-1)^p\left(\frac{1}{6}\,p + p^2 + \frac{4}{3}\,p^3 - \frac{1}{2}\,p^4\right)\ ,\\
\hat{\mathrm{n}}^0{}_{(3,3,3)} &= (-1)^p\left(\frac{1}{3}\,p + \frac{5}{6}\,p^2 + \frac{2}{3}\,p^3 + \frac{1}{6}\,p^4\right)\ .
\end{aligned} \tag{2.62}$$

Despite their appearance, one can easily check the integrality of the expressions (2.60)–(2.62) for any integer p. The quantity $\hat{\mathrm{n}}^0{}_{(0,n,0)}(p) = \mathrm{n}^0{}_n(p)$ computes the genus zero Gopakumar-Vafa invariants for closed topological strings on X_p. In particular, for $p = 1$ the threefold $X_1 = K_{\mathbb{P}^1}^{1/2} \oplus K_{\mathbb{P}^1}^{1/2}$ is the resolved conifold (with $K_{\mathbb{P}^1}$ the canonical line bundle over $\mathbb{P}^1$) which may be described using the large N geometric transition from $\mathrm{U}(N)$ Chern-Simons theory on S^3 [44]. In this case the Gopakumar-Vafa invariants are simply given by $\mathrm{n}^0{}_n(1) = -\delta_{n,1}$, which is the well-known result. Similarly, for $p = 2$ (and also $p = 0$) one has $X_2 = \mathcal{O} \oplus K_{\mathbb{P}^1}$ (with $\mathcal{O}$ the trivial line bundle over $\mathbb{P}^1$) and one finds the invariants $\mathrm{n}^0{}_n(2) = \delta_{n,1}$, again as expected [51]. These patterns can be further verified by extending the computations above up to seventh order (see appendix C). For $p \geq 3$ our calculations suggest that there are infinitely many non-vanishing Gopakumar-Vafa invariants of X_p. As we will see in section 4, this observation is related to the finite radius of convergence of the topological string perturbation series for $p > 2$ and it suggests an ultimate reason for the different phase structures of the theories with $p = 0, 1, 2$ and $p > 2$.

2.6 Gross-Taylor expansion as a topological string theory

We will now establish the connection between topological string theory and the Gross-Taylor string expansion. By specializing to the genus zero contributions, we will further establish the connection with the Crescimanno-Taylor [41] expansion which will be used extensively in the next section. Recall from section 2.2 that ordinary Yang-Mills theory on S^2 is reached in the limit where $p \to \infty$ with the area combination $g_s N\, p = A$ fixed. To describe this limit, we first rewrite the chiral partition function (2.14) in terms of the relevant parameters as

$$\mathcal{Z}^{q\mathrm{YM},0}_{\mathrm{chiral}} = \sum_{n=1}^{\infty} \mathrm{e}^{\,-(\frac{1}{2}-\frac{1}{p})n\,A} \sum_{\hat{R}} \mathrm{e}^{\,-\kappa_{\hat{R}} A/2\,N}\, W_{\hat{R}}\big(\,\mathrm{e}^{\,A/p\,N}\big)^2 \times \prod_{i=1}^{i_{\max}} \prod_{j=1}^{n_i} \Big(1-\,\mathrm{e}^{\,-\frac{A}{p\,N}\,(i-j)}\;\,\mathrm{e}^{\,-A/p}\Big)^2 \ . \qquad (2.63)$$

The crucial observation is that the quantum dimension (2.15) tends smoothly to the ordinary classical dimension $\dim(\hat{R})$ of the $\mathrm{SU}(N)$ representation $\hat{R}$ in the limit $g_s \to 0$. This means that, in the double limit $N \to \infty$, $p \to \infty$, the factor $\dim_q(\hat{R})^2$ must contain terms reproducing the expansion of the classical dimension

$$\dim\big(\hat{R}\big) = \xi_{2n}\, N^{2n} + \xi_{2n-1}\, N^{2n-1} + \dots \ . \qquad (2.64)$$

These terms correspond to the contributions

$$\dim_q\big(\hat{R}\big) = \xi_{2n} \left(\frac{p\,N}{A}\right)^{2n} \left(\frac{A}{p}\right)^{2n} + \xi_{2n-1} \left(\frac{p\,N}{A}\right)^{2n-1} \left(\frac{A}{p}\right)^{2n-1} + \dots \ . \qquad (2.65)$$

The subleading terms as $g_s \to 0$ are of the form $g_s^m\,(g_s N)^k = (\frac{A}{p\,N})^m\,(\frac{A}{p})^k$ with $m + k > 0$ by smoothness of the classical limit. From the explicit expression in (2.63) it follows that

$k \geq 0$. If $m < 0$ the corrections are $(\frac{p\,N}{A})^{|m|}\,(\frac{A}{p})^k = N^{|m|}\,(\frac{A}{p})^{k-|m|}$ and therefore correspond to $\frac{A}{p}$ corrections in (2.65) to the classical terms given by

$$\dim_q\,(\hat{R}) = \xi_{2n}\,N^{2n}\left(1+\alpha_{2n}\,\frac{A}{p}+\dots\right)+\xi_{2n-1}\,N^{2n-1}\left(1+\alpha_{2n-1}\,\frac{A}{p}+\dots\right)+\dots\,. \tag{2.66}$$

On the other hand, when $m > 0$ there are corrections of the type $\frac{1}{N^m}\,(\frac{A}{p})^{k+m}$ which correspond to both "quantum" contributions to the genus expansion and to $\frac{A}{p}$ corrections. These terms do not appear at leading order N^2 in the free energy.

We can now explicitly match the $p \to \infty$ limit of the chiral topological string expansion with the usual chiral Gross-Taylor [31] series at leading order. The genus zero chiral free energy of the topological string is given by

$$\mathcal{F}_0\big(t\,,\,\hat{t}\,;\,p\big) = \frac{1}{g_s^2}\sum_{n=1}^{\infty}\mathrm{e}^{\,-n\,t}\sum_{k=0}^{2n}\mathrm{e}^{\,-2\,k\,\hat{t}}\sum_{k_1=0}^{k}\hat{\mathrm{N}}^0{}_{(k_1,n,k-k_1)}(p) \tag{2.67}$$

in terms of Gromov-Witten invariants of $\hat{X}_p$. By rewriting this expansion in terms of QCD_2 parameters and expanding in $\frac{1}{p}$ we get

$$\mathcal{F}_0(A;p) = N^2\sum_{n=1}^{\infty}\mathrm{e}^{\,-n\,A/2}\sum_{k=0}^{2n}\sum_{l,m=1}^{\infty}\frac{k^l\,n^m}{l!\,m!}\left(-\frac{A}{p}\right)^{l+m-2}\sum_{k_1=0}^{k}\hat{\mathrm{N}}^0{}_{(k_1,n,k-k_1)}(p)\ . \tag{2.68}$$

On the other hand, the genus zero chiral free energy of the QCD_2 string can be written in the form

$$\Phi_0(A) = N^2\sum_{n=1}^{\infty}\mathrm{e}^{\,-n\,A/2}\sum_{j=0}^{2n-2}\eta_j^{(n)}\ A^j \tag{2.69}$$

since the branch point and Ω-point singularities of the holomorphic string maps generate, at winding number n, a polynomial of degree $2n-2$ in the area A. The matching of (2.68) and (2.69) thereby determines the polynomial coefficients $\eta_j^{(n)}$ in terms of the leading behaviour as $p \to \infty$ of Gromov-Witten invariants through

$$\eta_j^{(n)} = \frac{1}{(j+2)!}\,\lim_{p\to\infty}\,(-p)^{-j}\sum_{k=0}^{2n}(n+k)^{j+2}\sum_{k_1=0}^{k}\hat{\mathrm{N}}^0{}_{(k_1,n,k-k_1)}(p)\ . \tag{2.70}$$

This relationship suggests an explicit realization of two-dimensional chiral Yang-Mills theory as a topological string theory. From a physical perspective, as the degree of $\mathcal{O}(-p) \to \mathbb{P}^1$ grows the higher Kaluza-Klein modes of any section of this line bundle decouple and we may formally identify $\mathcal{O}(-\infty)$ with the trivial line bundle $\mathbb{P}^1 \times \mathbb{C}$. Thus the Gromov-Witten invariants of X_∞ formally reproduce the counting of holomorphic maps into the base sphere. We will see some explicit examples of this in section 4. Note that the D-brane insertions in the fibers of the original fibration (2.1) play a crucial role in this identification.

Let us now describe the mathematical implications of the relationship (2.70). Let $\overline{M}_g(\hat{X}_p,\boldsymbol{n})$ be the Deligne-Mumford moduli space of stable holomorphic maps from connected genus g curves to $\hat{X}_p$ which represent the class $\boldsymbol{n} \in H_2(\hat{X}_p, \mathbb{Z})$. Then the Gromov-Witten invariants of $\hat{X}_p$ are given by

$$\hat{\mathrm{N}}^g{}_{\boldsymbol{n}}(p) = \int_{\overline{M}_g(\hat{X}_p, \boldsymbol{n})} 1 \ . \tag{2.71}$$

More precisely, the integral should be evaluated over the virtual fundamental class of the moduli space of maps. With the appropriate push-forward map one can use virtual localization techniques to reduce the integral (2.71) to Hodge integrals over the moduli space of curves with n punctures $\overline{M}_{g,n}$ of dimension $3g-3+n$. While this is always possible to do in principle, in practise it is quite difficult.

On the other hand, the relationship (2.70) gives such a reduction in the limit $p \to \infty$ by relating (2.71) to the Gromov-Witten theory of the base $\mathbb{P}^1$. For this, let $\mathrm{H}^g{}_{\vec{\mu}}$ be the Hurwitz numbers of $\mathbb{P}^1$ corresponding to the partition $\vec{\mu} = (1^{\mu_1}\, 2^{\mu_2} \cdots)$, i.e. the number of genus g branched covering maps to $\mathbb{P}^1$ with ramification $\vec{\mu}$ over ∞ and simple ramification over $\mathbb{P}^1 \setminus \infty$. These integers can be represented as integrals over the moduli space $\overline{M}_g(\mathbb{P}^1, n)$ of holomorphic maps of genus g and winding number n to $\mathbb{P}^1$. Let us illustrate how this works explicitly in the simplest case of the trivial partition $\vec{\mu} = (1^n)$, i.e. the case of genus g simple branched covers of degree n over $\mathbb{P}^1$. By the Riemann-Hurwitz theorem, such maps have $r = 2g-2+2n$ simple ramification points. There is a natural map $\beta : \overline{M}_g(\mathbb{P}^1, n) \to \mathbb{P}^r$ which assigns to each map its branch point locus. Let Ξ be the hyperplane class of $\mathbb{P}^r$ associated to the canonical hyperplane bundle. Then the simple Hurwitz numbers can be represented as Gromov-Witten integrals [52]–[54]

$$\mathrm{H}^g{}_n := \mathrm{H}^g{}_{(1^n)} = \int_{\overline{M}_g(\mathbb{P}^1, n)} \beta^*(\Xi) \ . \tag{2.72}$$

The virtual localization formula may now be used to compute the integral (2.72). The standard action of the multiplicative group of complex numbers $\mathbb{C}^\times$ on $\mathbb{P}^1$ induces a $\mathbb{C}^\times$-action on the moduli space $\overline{M}_g(\mathbb{P}^1, n)$ for which the pullback $\beta^*(\Xi)$ is an equivariant class. The fixed points of this group action are products of moduli spaces of n-punctured curves $\overline{M}_{g,n}$. The localization formula thereby reduces (2.72) to tautological intersection indices on $\overline{M}_{g,n}$. The result can be expressed as a Hodge integral as follows. Let $\mathcal{L}_i$, $i = 1, \dots, n$ be the canonical line bundles over $\overline{M}_{g,n}$, and define tautological classes as the first Chern classes $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{M}_{g,n}, \mathbb{Q})$. Let $\mathcal{E}_g \to \overline{M}_{g,n}$ be the rank g Hodge bundle, and denote the corresponding Chern classes by $\lambda_k = c_k(\mathcal{E}_g) \in H^{2k}(\overline{M}_{g,n}, \mathbb{Q})$. With $\lambda_0 := 1$, the localization formula then reads [52]–[54]

$$\mathrm{H}^g{}_n = \frac{(2g-2+2n)!}{n!} \sum_{k=0}^{g} (-1)^k \int_{\overline{M}_{g,n}} \lambda_k \wedge \bigwedge_{i=1}^{n} (1-\psi_i)^{-1} \tag{2.73}$$

for $(g,n) \neq (0,1), (0,2)$.

On the other hand, the coefficient of $A^{2n-2}/(2n-2)!$ in the area polynomials of (2.69) is precisely the number of topologically inequivalent holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ with $2n-2$ branch point singularities of fixed image and no Ω-point singularities. Thus $\mathrm{H}^0{}_n =$

$(2n-2)!\,\eta^{(n)}_{2n-2}$. Using (2.70) we may thereby write a large p localization formula for the Gromov-Witten invariants (2.71) as

$$\lim_{p\to\infty}\frac{1}{p^{2n-2}}\sum_{k=0}^{2n}(n+k)^{2n}\sum_{k_1=0}^{k}\int_{\overline{M}_0\left(\hat{X}_p\,,\,(k_1,n,k-k_1)\right)}1=\frac{(2n)!}{n!}\int_{\overline{M}_{0,n}}\bigwedge_{i=1}^{n}(1-\psi_i)^{-1} \quad (2.74)$$

for $n\neq 1,2$. The right-hand side of eq. (2.74) can be evaluated explicitly in these simple instances in terms of the Hodge integrals $\sum_{a_k}\int_{\overline{M}_{0,n}}\psi_1^{\wedge a_1}\wedge\cdots\wedge\psi_n^{\wedge a_n}$, giving the anticipated genus zero Hurwitz formula ${\rm H^0}_n=(2n-2)!\,n^{n-3}/n!$.

With some work one can extend these identifications to both higher genera ($g>0$) and to covering maps with non-simple ramifications of their branch point singularities ($j<2n-2+2g$). In these cases one must take into account the contributions from Ω-point singularities in the above analysis. In the chiral gauge theory, the singularity at an Ω-point is a multiple branch point singularity which is described by an arbitrary permutation on the sheets of the covering space when following the lift of a closed target space curve on $\mathbb{P}^1$ around the Ω-point. These additional singularities feature in nicely in what is known about the evaluation of the corresponding localization integrals in Gromov-Witten theory. Any continuous mapping from a Riemann surface to $\mathbb{P}^1$ is, up to homotopy, the composition of a pinch map (collapsing regions of the surface to a single point) and a branched covering map. The pinch maps are responsible for the Ω-point singularities and are related to appearance of *multiple* Hurwitz numbers (ramified over more than one point than just ∞ on $\mathbb{P}^1$) in the computation of the Gromov-Witten invariants of the original threefold X_p [33]. Physically, they are directly related to the insertions of the fiber D-branes in the topological string theory on X_p. We will return to the relationship between Hurwitz numbers and Gromov-Witten invariants in section 4.

3. Matrix model formalism

In this section we shall investigate the large N limit of the chiral sector of q-deformed Yang-Mills theory on the sphere S^2 by means of matrix model techniques. From a detailed analysis of the resulting saddle-point equation we will obtain the phase structure of the gauge theory at large N. We then show that in an appropriate strong-coupling phase the chiral topological string theory of the previous section is recovered from the large N gauge dynamics.

3.1 Saddle point solution

The saddle point equation governing the distribution of Young tableaux variables in chiral q-deformed Yang-Mills theory coincides with that of the full coupled gauge theory [34, 38, 39]. The new information is completely encoded in boundary conditions on the solutions to this equation. In the chiral sector we sum not over all representations but only over those which have finitely many "large" numbers of rows. With $x_i:=\frac{i}{N}$, this means that the lengths of the rows satisfy the constraints

$$-\frac{n_i}{N}+x_i-\frac{1}{2}=0\ ,\quad i\geq k \quad (3.1)$$

for some k such that $x_k \to c_0 \in \mathbb{R}$ as $N \to \infty$. Equivalently, in the large N limit we can characterize the chiral theory by setting

$$n(x) = x - \frac{1}{2} \,, \quad x \geq c_0 \tag{3.2}$$

where $n(x)$ is the number of boxes at position x which is a monotonic function with $n(x_i) = \frac{n_i}{N}$. In terms of the distribution function $\rho(x)$ for the Young tableaux, this constraint can be written as

$$\rho(n) := \frac{\mathrm{d}x(n)}{\mathrm{d}n} = 1 \,, \quad c = c_0 - \frac{1}{2} \leq n \leq \frac{1}{2} \,. \tag{3.3}$$

In the matrix model approach we therefore have to solve the saddle-point equation [34, 38, 39]

$$\frac{A\,z}{2} = \hat{t} \int_b^{1/2} \mathrm{d}w \; \rho(w) \; \coth\big(\hat{t}\,(z-w)\big) \,, \tag{3.4}$$

where $A = 2\,\hat{t}\,p = N\,g_s p$ is the area parameter introduced previously and the boundary conditions for $\rho(z)$ are depicted in figure 5.

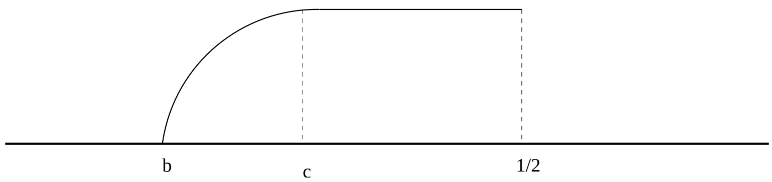

Figure 5: Ansatz for the distribution function $\rho(z)$.

Since $\rho(z) = 1$ for $z \in [c, \frac{1}{2}]$, eq. (3.4) can be cast in the more standard form

$$\frac{A\,z}{2} = \hat{t} \int_b^{c} \mathrm{d}w \; \tilde{\rho}(w) \; \coth\big(\hat{t}\,(z-w)\big) - \log\left|\frac{\sinh\big(\hat{t}\,(z-\frac{1}{2})\big)}{\sinh\big(\hat{t}\,(z-c)\big)}\right| \tag{3.5}$$

where $\tilde{\rho} := \rho|_{[b,c]}$. The new boundary conditions are now translated into a c-dependent modification of the original potential. This apparently mild modification is similar to the one which occurs in the two-cut solution of the full non-chiral theory and it will play an important role in recovering the correct perturbative string expansion. To solve eq. (3.5), we first reduce it to a classical Riemann-Hilbert problem by changing variables from w and z to $s = \mathrm{e}^{\,2\,\hat{t}\,w + 8\,\hat{t}^{\,2} c/A}$ and $u = \mathrm{e}^{\,2\,t\,z + 8\,\hat{t}^{\,2} c/A}$ to get

$$\frac{A}{8\,\hat{t}^{\,2}}\,\frac{\log(s)}{s} + \frac{2}{\hat{t}\,s}\log\left|\frac{s - \mathrm{e}^{\,\hat{t} + 8\,\hat{t}^{\,2}c/A}}{s - \mathrm{e}^{\,2\,\hat{t}\,c + 8\,\hat{t}^{\,2}c/A}}\right| = \int_{\mathrm{e}^{\,2\,\hat{t}\,b + 8\,\hat{t}^{\,2}c/A}}^{\mathrm{e}^{\,2\,\hat{t}\,c + 8\,\hat{t}^{\,2}c/A}} \mathrm{d}u \; \frac{\varrho(u)}{s-u} \,, \tag{3.6}$$

where

$$\varrho(u) := \frac{\tilde{\rho}\big(\log(\,\mathrm{e}^{\,-8\,\hat{t}^{\,2}c/A}\,u)/2\,\hat{t}\,\big)}{2\,\hat{t}\,u} \,. \tag{3.7}$$

Instead of the pair $(A, \hat{t}\,)$ we shall use the variables (A, p), as this will enable a simpler comparison with the Gross-Taylor series later on. We also introduce the parameters

$$b' = \frac{A}{p^2}\,(p\,b + 2\,c) \,, \quad c' = \frac{A\,c}{p^2}\,(p+2) \,, \quad d' = \frac{2\,A}{p^2}\left(c + \frac{p}{4}\right) \tag{3.8}$$

in order to simplify notation.

The solution to the integral equation (3.6) can then be written in terms of the corresponding resolvent function as

$$
\begin{aligned}
\omega(z) := \int_{\mathrm{e}^{b'}}^{\mathrm{e}^{c'}} \mathrm{d}u \, \frac{\varrho(u)}{z-u} &= \frac{p^2}{4\pi\,\mathrm{i}\,A}\sqrt{(z-\mathrm{e}^{c'})(z-\mathrm{e}^{b'})} \\
&\quad \times \oint_C \frac{\mathrm{d}w}{w(w-z)} \, \frac{\log(w)+\frac{2}{p}\,\log\left|\frac{w-\mathrm{e}^{d'}}{w-\mathrm{e}^{c'}}\right|}{\sqrt{(w-\mathrm{e}^{b'})(\mathrm{e}^{c'}-w)}} \,, \qquad (3.9)
\end{aligned}
$$

where the closed contour C encircles the support $[\mathrm{e}^{b'},\,\mathrm{e}^{c'}]$ of the distribution $\varrho(u)$ with counterclockwise orientation in the complex z-plane. If we choose the square root and logarithmic branch cuts in (3.9) as indicated in figure 6, then since the integrand decays as

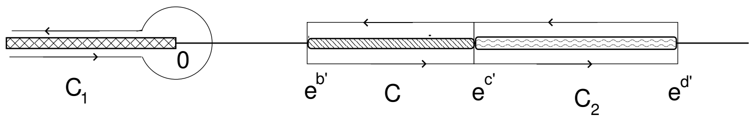

Figure 6: The contour C_1 surrounds the branch cut $(-\infty,0]$ of $\log(z)$, C_2 encircles the cut $[\mathrm{e}^{c'},\,\mathrm{e}^{d'}]$ of $\log\big(\frac{z-\mathrm{e}^{d'}}{z-\mathrm{e}^{c'}}\big)$, while C encloses the physical branch cut $[\mathrm{e}^{b'},\,\mathrm{e}^{c'}]$ of $\sqrt{(z-\mathrm{e}^{b'})(z-\mathrm{e}^{c'})}$.

w^{-3} at $|w|\to\infty$ we can deform the contour of integration C so that it encircles the cuts of the two logarithms. This deformation picks up an additional contribution from the pole at $w=z$ and we find

$$
\begin{aligned}
\omega(z) &= -\frac{p^2}{4\pi\,\mathrm{i}\,A}\sqrt{(z-\mathrm{e}^{c'})(z-\mathrm{e}^{b'})}\left[\oint_{C_1}\frac{\mathrm{d}w}{w(w-z)}\,\frac{\log(w)+\frac{2}{p}\,\log\Big(\frac{w-\mathrm{e}^{d'}}{w-\mathrm{e}^{c'}}\Big)}{\sqrt{(w-\mathrm{e}^{c'})(w-\mathrm{e}^{b'})}}\right. \\
&\quad \left. + \oint_{C_2}\frac{\mathrm{d}w}{w(w-z)}\,\frac{\log(w)+\frac{2}{p}\,\log\Big(\frac{w-\mathrm{e}^{d'}}{w-\mathrm{e}^{c'}}\Big)}{\sqrt{(w-\mathrm{e}^{c'})(w-\mathrm{e}^{b'})}}\right] - \frac{p^2}{2\,A}\,\frac{\log(z)}{z} - \frac{p}{A\,z}\,\log\left(\frac{z-\mathrm{e}^{d'}}{z-\mathrm{e}^{c'}}\right) \\
&= -\frac{p^2}{2\,A}\sqrt{(z-\mathrm{e}^{c'})(z-\mathrm{e}^{b'})}\left[\int_{-\infty}^{-\epsilon}\frac{\mathrm{d}w}{w(w-z)}\,\frac{1}{\sqrt{(\mathrm{e}^{c'}-w)(\mathrm{e}^{b'}-w)}}\right. \\
&\quad \left. + \frac{\log(\epsilon)}{z}\,\mathrm{e}^{-(b'+c')/2} - \frac{2}{p}\int_{\mathrm{e}^{c'}}^{\mathrm{e}^{d'}}\frac{\mathrm{d}w}{w(w-z)}\,\frac{1}{\sqrt{(w-\mathrm{e}^{c'})(w-\mathrm{e}^{b'})}}\right] \qquad (3.10) \\
&\quad -\frac{p^2}{2\,A}\,\frac{\log(z)}{z} - \frac{p}{A\,z}\,\log\left(\frac{z-\mathrm{e}^{d'}}{z-\mathrm{e}^{c'}}\right) - \frac{p}{Az}\,(d'-c')\,\mathrm{e}^{-(b'+c')/2}\sqrt{(z-\mathrm{e}^{c'})(z-\mathrm{e}^{b'})},
\end{aligned}
$$

where $\epsilon\to0^+$ and we have chosen positive square roots for the integrands over the real line.

The required integrals are computed in appendix D and one arrives at the resolvent

$$
\begin{aligned}
\omega(z) = -\frac{p}{2\,A\,s}\Bigg\{ & 2\,p\,\log\left(\frac{e^{\,b'/2}\,\sqrt{z-e^{\,c'}}+e^{\,c'/2}\,\sqrt{z-e^{\,b'}}}{\sqrt{z-e^{\,c'}}+\sqrt{z-e^{\,b'}}}\right) \\
& +p\;e^{-(b'+c')/2}\sqrt{(z-e^{\,c'})(z-e^{\,b'})}\,\left(b'+c'-2\,\log\Big(\frac{e^{\,b'/2}+e^{\,c'/2}}{2}\Big)\right) \\
& +2\,\log\left(\frac{\left(\sqrt{e^{\,d'}-e^{\,b'}}+\sqrt{\frac{z-e^{\,b'}}{z-e^{\,c'}}}\,(e^{\,d'}-e^{\,c'})\right)^2}{e^{\,c'}-e^{\,b'}}\right) \\
& +2\;e^{-(b'+c')/2}\sqrt{(z-e^{\,c'})(z-e^{\,b'})} \\
& \times\left[\log\left(\frac{\left(e^{\,c'/2}\sqrt{e^{\,d'}-e^{\,b'}}+e^{\,b'/2}\sqrt{e^{\,d'}-e^{\,c'}}\right)^2}{e^{\,c'}-e^{\,b'}}\right)-c'\right]\Bigg\}\,.
\end{aligned}
\tag{3.11}
$$

The asymptotic boundary condition for $\omega(z)$ is fixed by the normalization of the spectral density to be

$$
\lim_{|z|\to\infty}\,\omega(z)=\frac{1}{z}\int_{e^{\,b'}}^{e^{\,c'}}du\;\varrho(u)=\frac{\frac{1}{2}+c}{z}\;, \tag{3.12}
$$

which on comparison with the asymptotic behaviour of (3.11) yields a pair of equations for the unknown parameters b and c of the saddle point solution. The constant asymptotic value of (3.11) at infinity is given by all terms multiplying the square root $\sqrt{(z-e^{\,c'})(z-e^{\,b'})}$. Requiring them to vanish imposes the boundary condition

$$
\frac{p}{2}\left(b'+c'-2\,\log\Big(\frac{e^{\,b'/2}+e^{\,c'/2}}{2}\Big)\right)+\log\left(\frac{\left(e^{\,c'/2}\sqrt{e^{\,d'}-e^{\,b'}}+e^{\,b'/2}\sqrt{e^{\,d'}-e^{\,c'}}\right)^2}{e^{\,c'}-e^{\,b'}}\right)=c'\,. \tag{3.13}
$$

Extracting the subleading behaviour of (3.11) at $|z|\to\infty$ requires just a bit more effort and is fixed by eq. (3.12) to be

$$
p\,\log\left(\frac{e^{\,b'/2}+e^{\,c'/2}}{2}\right)+\log\left(\frac{\left(\sqrt{e^{\,d'}-e^{\,b'}}+\sqrt{e^{\,d'}-e^{\,c'}}\right)^2}{e^{\,c'}-e^{\,b'}}\right)=-\frac{A}{p}\left(\frac{1}{2}+c\right)\;, \tag{3.14}
$$

where we have dropped a term which vanishes by eq. (3.13).

When the equations above for the endpoints of the support interval hold, all the terms in $\omega(z)$ which are proportional to $\sqrt{(z-e^{\,c'})(z-e^{\,b'})}$ vanish identically, and thus the resolvent function (3.11) assumes the very simple and compact form

$$
\omega(z)=-\frac{p^2}{A\,z}\,\log\left(\frac{e^{\,b'/2}\,\sqrt{z-e^{\,c'}}+e^{\,c'/2}\,\sqrt{z-e^{\,b'}}}{\sqrt{z-e^{\,c'}}+\sqrt{z-e^{\,b'}}}\right)
$$

$$-\,\frac{p}{A\,z}\,\log\left(\frac{\left(\sqrt{\mathrm{e}^{\,d'}-\mathrm{e}^{\,b'}}+\sqrt{\frac{z-\mathrm{e}^{\,b'}}{z-\mathrm{e}^{\,c'}}}\,(\,\mathrm{e}^{\,d'}-\mathrm{e}^{\,c'})\right)^2}{\mathrm{e}^{\,c'}-\mathrm{e}^{\,b'}}\right)\ . \tag{3.15}$$

The distribution function (3.7) is then determined by the jump discontinuity of (3.15) across the cut $[\,\mathrm{e}^{\,b'},\,\mathrm{e}^{\,c'}]$ as

$$\begin{aligned}\varrho(s) &= \frac{\omega(s+\mathrm{i}\,\epsilon)-\omega(s-\mathrm{i}\,\epsilon)}{2\pi\,\mathrm{i}}\\ &= \frac{p^2}{\pi\,A\,s}\,\arctan\left(\sqrt{\frac{\mathrm{e}^{\,c'}-s}{s-\mathrm{e}^{\,b'}}}\right)-\frac{p^2}{\pi\,A\,s}\,\arctan\left(\mathrm{e}^{\,b'/2-c'/2}\sqrt{\frac{\mathrm{e}^{\,c'}-s}{s-\mathrm{e}^{\,b'}}}\right)\\ &\quad+\frac{2\,p}{\pi\,A\,s}\,\arctan\left(\sqrt{\frac{s-\mathrm{e}^{\,b'}}{\mathrm{e}^{\,c'}-s}\,\frac{\mathrm{e}^{\,d'}-\mathrm{e}^{\,c'}}{\mathrm{e}^{\,d'}-\mathrm{e}^{\,b'}}}\right)\ .\end{aligned} \tag{3.16}$$

This distribution manifestly satisfies the required boundary conditions, since $\varrho(\,\mathrm{e}^{\,c'})=p/A\,s$ and $\varrho(\,\mathrm{e}^{\,b'})=0$.

3.2 Crescimanno-Taylor equations

Eqs. (3.13) and (3.14) encode most of the gauge dynamics of the large N limit. Before attempting a systematic solution, we shall verify that they are consistent with the Crescimanno-Taylor equations for the QCD_2 string [41], which will also provide a non-trivial check of our equations. The usual chiral gauge theory on S^2 should emerge in our framework when $p\to\infty$ with the area A of the sphere fixed. By substituting in the large p expansions

$$\begin{aligned}\sqrt{\mathrm{e}^{\,d'}-\mathrm{e}^{\,c'}} &= \left(\frac{A}{p}\right)^{1/2}\sqrt{\frac{1}{2}-c}+\dots\ ,\\ \sqrt{\mathrm{e}^{\,d'}-\mathrm{e}^{\,b'}} &= \left(\frac{A}{p}\right)^{1/2}\sqrt{\frac{1}{2}-b}+\dots\ ,\\ \mathrm{e}^{\,c'}-\mathrm{e}^{\,b'} &= \frac{A}{p}\,(c-b)+\dots\ ,\\ \log\left(\frac{\mathrm{e}^{\,b'/2}+\mathrm{e}^{\,c'/2}}{2}\right) &= \frac{A}{2\,p}\,(b+c)+\dots\end{aligned} \tag{3.17}$$

the expression (3.13) takes the form

$$-\frac{A}{4}\,(b+c)=\log\left(\frac{\left(\sqrt{\frac{1}{2}-c}+\sqrt{\frac{1}{2}-b}\right)^2}{(c-b)}\right)\ . \tag{3.18}$$

This is the first Crescimanno-Taylor equation. Performing the same expansion in eq. (3.14) at leading order again recovers eq. (3.18).

The second Crescimanno-Taylor equation appears by expanding either of eqs. (3.13) or (3.14) to second order in $\frac{1}{p}$, giving the same result

$$\frac{A}{16}\,(b-c)^2 + \sqrt{\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)} = 1\ . \tag{3.19}$$

This equation appears only at second order in $\frac{1}{p}$ because we have summed the equations with a weight depending on p in order to have the simplest possible expressions. This procedure mixes the various orders of the expansion.

3.3 Phase transitions

The problem of the existence of solutions to eqs. (3.13) and (3.14) is most easily addressed by introducing a new set of variables that are centered around the point $\frac{1}{2}$, given by the Kähler modulus $t = \frac{A}{2p}\,(p-2)$ and the new endpoint parameters

$$\hat{b} = d' - b' = \frac{2\,t}{p-2}\left(\frac{1}{2}-b\right)\ ,\quad \hat{c} = d' - c' = \frac{2\,t}{p-2}\left(\frac{1}{2}-c\right)\ . \tag{3.20}$$

In terms of these parameters eq. (3.13) takes the form

$$-p\,\log\left(\frac{\mathrm{e}^{\,\hat{b}/2}+\mathrm{e}^{\,\hat{c}/2}}{2}\right) + \log\left(\frac{\left(\sqrt{\mathrm{e}^{\,\hat{b}}-1}+\sqrt{\mathrm{e}^{\,\hat{c}}-1}\right)^2}{\mathrm{e}^{\,\hat{b}}-\mathrm{e}^{\,\hat{c}}}\right) = -\frac{t}{2}\,\frac{p+2}{p-2}\ , \tag{3.21}$$

while eq. (3.14) becomes

$$p\,\log\left(\frac{\mathrm{e}^{\,-\hat{b}/2}+\mathrm{e}^{\,-\hat{c}/2}}{2}\right) + \log\left(\frac{\left(\sqrt{1-\mathrm{e}^{\,-\hat{b}}}+\sqrt{1-\mathrm{e}^{\,-\hat{c}}}\right)^2}{\mathrm{e}^{\,-\hat{c}}-\mathrm{e}^{\,-\hat{b}}}\right) = \frac{t}{2}\ . \tag{3.22}$$

In principle one should now fix the coupling constant t and solve for the endpoints $\hat{b}$ and $\hat{c}$. However, this approach is not the most practical one for a numerical analysis. Thus we choose instead to fix $x = \mathrm{e}^{\,-\hat{c}/2} \in [0,1]$ and determine the corresponding variables $y = \mathrm{e}^{\,-\hat{b}/2} \in [0,1]$ and t. From eq. (3.22) we have

$$t = 2\,p\,\log\left(\frac{x+y}{2}\right) + 4\,\log\left(\frac{\sqrt{1-x^2}+\sqrt{1-y^2}}{\sqrt{x^2-y^2}}\right)\ , \tag{3.23}$$

and substituting this into eq. (3.21) yields

$$T(x,y;p) := -1 + \frac{4^{\frac{p}{p-2}}\,x\,y\,\left(x^2-y^2\right)^{\frac{4}{(p-2)\,p}}\left(x\,\sqrt{1-y^2}+y\,\sqrt{1-x^2}\,\right)^{\frac{2}{p}}}{(x+y)^{\frac{2\,p}{p-2}}\left(\sqrt{1-x^2}+\sqrt{1-y^2}\,\right)^{\frac{2(2+p)}{(p-2)\,p}}} = 0\ . \tag{3.24}$$

Once we have fixed the geometrical datum p and given our choice for x, the algebraic equation (3.24) determines y and eq. (3.23) then gives the Kähler modulus t.

When we take into account the constraint $y \leq x$ (i.e. $c \geq b$) eq. (3.24) does not always admit a solution for a given p. Generally, the interval $[0,1] = I_1 \amalg I_2 \amalg I_3$ decomposes into three disjoint, connected subintervals in the following manner. For $x \in I_1$ there are two solutions y of eq. (3.24), for $x \in I_2$ there are no solutions, and for $x \in I_3$ there are again two solutions. The sizes of the of these subintervals depend on p. The region I_1 increases as p grows, while I_2 and I_3 decrease very rapidly. A representative example of this behaviour is depicted in figure 7. We also have to impose the additional constraint on the spectral

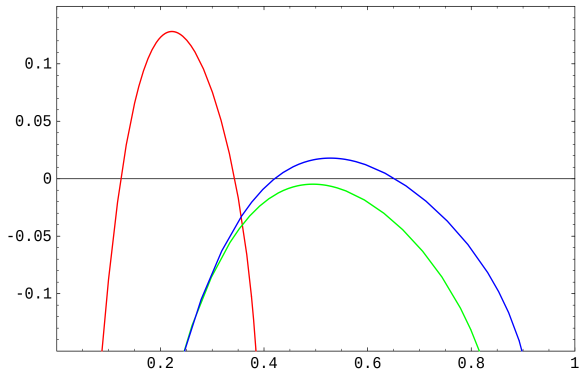

Figure 7: Plot of $T(x, y; p)$ as a function of y for $p = 7$ and $x = 0.4 \in I_1$ (left curve), $x = 0.89 \in I_2$ (center curve) and $x = 0.95 \in I_3$ (right curve).

distribution that $\rho \leq 1$. This requirement immediately rejects the solution in the interval I_1 for which the density has the form depicted in figure 8.

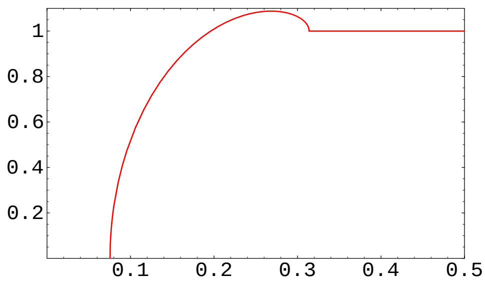

Figure 8: Unphysical solution $\rho(s)$ versus s for $x = 0.4 \in I_1$.

A clearer picture of the situation is obtained if we draw the phase diagram in the (c, t)-plane for various values of the geometrical parameter p (figure 9). The qualitative behaviour is very similar to that of ordinary chiral QCD$_2$ [41]. The lines coming from the region of large t all represent physical solutions up to a certain critical value $t_c^+(p)$. This means that the one-cut solution allows us to explore the region with large Kähler parameter. When the lines reach and go below $t = t_c^+(p)$, the distribution function ρ becomes larger than 1 and a phase transition occurs. Rather remarkably, the value $t_c^+(p)$ is very close to the value of the Kähler modulus that triggers the phase transition in the full coupled

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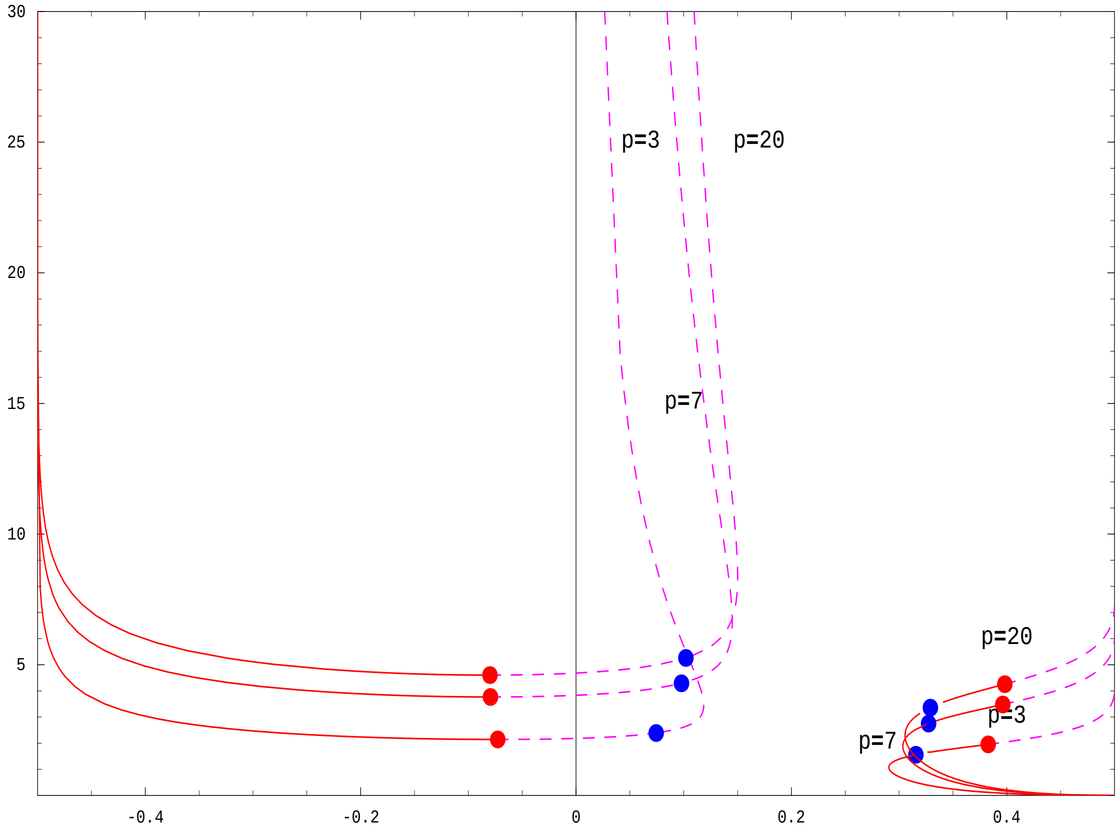

Figure 9: Behaviour of t as a function of c for $p = 3, 7, 20$. The dashed parts of the curves indicate the unphysical regions having $\rho > 1$. The dots separating a dashed line from a solid line represent the critical points of a phase transition. The other dots represent diramation points for the solutions of the equation $T(x, y; p) = 0$ for each branch.

q-deformed gauge theory [34, 38, 39]. At this point no physical solution exists until we reach a second critical point $t_c^-(p)$. This point connects the line coming from the region of small t. The distribution function ρ is again smaller than 1 for $0 < t < t_c^-(p)$ and a second phase transition occurs at $t = t_c^-(p)$. We conclude that our one-cut solution describes the large N gauge theory in two distinct phases, one which covers the large values of the Kähler parameter t and the other describing the small values of t. To connect these two phases, one would have to construct an appropriate two-cut solution in the intermediate region.

3.4 Topological strings in the strong-coupling phase

We expect that the topological string theory on X_p will emerge from the q-deformed gauge dynamics when t is large, i.e. for $t > t_c^+(p)$ where the one-cut solution constructed above is valid. We therefore seek a consistent expansion of the solution of the saddle-point equation for large values of the coupling constant t. For this, we assume that the endpoints b and c are finite in the limit $t \to \infty$, or equivalently that $\hat{b}$ and $\hat{c}$ diverge at most linearly in t.

Then eqs. (3.13) and (3.14) reduce for large t to

$$\hat{b} = \hat{c} = \frac{2\,t}{p-2} = 2\,\hat{t}\ , \tag{3.25}$$

or equivalently $b = c = -\frac{1}{2}$. This means that at strong-coupling the distribution function $\rho(z)$ tends to become flat and symmetric about the origin. The particular form of the saddle point equation (3.5) suggests that corrections to this result are exponentially suppressed in t. Thus we look for a solution of the form

$$\begin{aligned} \hat{b} &= 2\,\hat{t} + \sum_{n=1}^{\infty} r_n\ \tau^n\ , \\ \hat{c} &= 2\,\hat{t} + \sum_{n=1}^{\infty} s_n\ \tau^n \end{aligned} \tag{3.26}$$

where $\tau := \mathrm{e}^{-\xi\,t/2}$. Imposing consistency of our ansatz fixes the parameter $\xi = 1$.

Proceeding iteratively, we arrive at

$$\begin{aligned} \hat{b} = 2\,\hat{t} &+ \Big[2 - 2\ \mathrm{e}^{-2\,\hat{t}}\Big]\ \mathrm{e}^{-t/2} + \Big[1 - p + 2\,p\ \mathrm{e}^{-2\,\hat{t}} - (1+p)\ \mathrm{e}^{-4\,\hat{t}}\Big]\ \mathrm{e}^{-t} \\ &+ \bigg[\frac{2}{3} - 2\,p + p^2 + \big(2\,p - 3\,p^2\big)\ \mathrm{e}^{-2\,\hat{t}} + \big(2\,p + 3\,p^2\big)\ \mathrm{e}^{-4\,\hat{t}} \\ &- \Big(\frac{2}{3} + 2\,p + p^2\Big)\ \mathrm{e}^{-6\,\hat{t}}\bigg]\ \mathrm{e}^{-3\,t/2} + O\left(\mathrm{e}^{-2\,t}\right)\ , \\ \hat{c} = 2\,\hat{t} &- \Big[2 - 2\ \mathrm{e}^{-2\,\hat{t}}\Big]\ \mathrm{e}^{-t/2} + \Big[1 - p + 2\,p\ \mathrm{e}^{-2\,\hat{t}} - (1+p)\ \mathrm{e}^{-4\,\hat{t}}\Big]\ \mathrm{e}^{-t} \\ &- \bigg[\frac{2}{3} - 2\,p + p^2 + \big(2\,p - 3\,p^2\big)\ \mathrm{e}^{-2\,\hat{t}} + \big(2\,p + 3\,p^2\big)\ \mathrm{e}^{-4\,\hat{t}} \\ &- \Big(\frac{2}{3} + 2\,p + p^2\Big)\ \mathrm{e}^{-6\,\hat{t}}\bigg]\ \mathrm{e}^{-3\,t/2} + O\left(\mathrm{e}^{-2\,t}\right)\ . \end{aligned} \tag{3.27}$$

Note that the corrections are order by order polynomials in $\mathrm{e}^{-2\,\hat{t}}$. This is very different from what happens in the full coupled theory where the corrections are given by infinite series. If we identify the power of $\mathrm{e}^{-t/2}$ with the winding number of the topological string expansion, then the behaviour of our solutions is exactly that expected from string theory. We recall from section 2.2 that the large N expansion of the partition function was organized in exactly the same way.

Our next goal is to compute the chiral partition function for large values of the Kähler parameter t. For this, we will calculate the derivative of the free energy with respect to the area A at fixed $\hat{t} = \frac{g_s N}{2}$. We have [34]

$$\frac{\partial \mathcal{F}_0\big(A\,,\,\hat{t}\,;\,p\big)}{\partial A} = \frac{1}{2}\int_b^{1/2} \mathrm{d}x\ x^2\ \rho(x) = \frac{1}{2}\int_b^{c} \mathrm{d}x\ x^2\ \rho(x) + \frac{1}{2}\left(\frac{1}{24} - \frac{c^3}{3}\right) \tag{3.28}$$

and so we need to compute the integral

$$\mathcal{I} := \frac{1}{2}\int_b^{c} \mathrm{d}x\ x^2\ \rho(x)\ . \tag{3.29}$$

We change variable $x = p\log(\mathrm{e}^{-2Ac/p^2}u)/A$ to get

$$\mathcal{I} = \frac{p^2}{2A^2}\int_{\mathrm{e}^{b'}}^{\mathrm{e}^{c'}} \mathrm{d}u\ \log^2\left(\mathrm{e}^{-2Ac/p^2}u\right)\varrho(u)\ . \tag{3.30}$$

To express the result in terms of the endpoint parameters $\hat{b}$ and $\hat{c}$, we set $u = s\ \mathrm{e}^{d'}$ and use the explicit solution (3.16) to obtain

$$\mathcal{I} = \frac{p^4}{2\pi A^3}\int_{\mathrm{e}^{-\hat{b}}}^{\mathrm{e}^{-\hat{c}}}\frac{\mathrm{d}s}{s}\ \log^2\left(\mathrm{e}^{A/2p}s\right)\left[\arctan\left(\sqrt{\frac{\mathrm{e}^{-\hat{c}}-s}{s-\mathrm{e}^{-\hat{b}}}}\right) - \arctan\left(\mathrm{e}^{(\hat{c}-\hat{b})/2}\sqrt{\frac{\mathrm{e}^{-\hat{c}}-s}{s-\mathrm{e}^{-\hat{b}}}}\right)\right.$$
$$\left. + \frac{2}{p}\arctan\left(\sqrt{\frac{s-\mathrm{e}^{-\hat{b}}}{\mathrm{e}^{-\hat{c}}-s}\,\frac{1-\mathrm{e}^{-\hat{c}}}{1-\mathrm{e}^{-\hat{b}}}}\right)\right]. \tag{3.31}$$

To simplify the analysis, it is convenient to integrate by parts to get

$$\mathcal{I} = \frac{c^3}{6} - \frac{1}{96\pi A^3\sqrt{1-\mathrm{e}^{-\hat{c}}}}\int_{\mathrm{e}^{-\hat{b}}}^{\mathrm{e}^{-\hat{c}}}\frac{\mathrm{d}s}{s\,(s-1)\sqrt{(\mathrm{e}^{-\hat{c}}-s)(s-\mathrm{e}^{-\hat{b}})}}$$
$$\times\left[\mathrm{e}^{-(\hat{b}+\hat{c})/2}\sqrt{1-\mathrm{e}^{-\hat{c}}}\,p\,(1-s) - 2\ \mathrm{e}^{-\hat{c}}\sqrt{1-\mathrm{e}^{-\hat{b}}}\,s\right.$$
$$\left. + \left(2\sqrt{1-\mathrm{e}^{-\hat{b}}} + \sqrt{1-\mathrm{e}^{-\hat{c}}}\,p\,(s-1)\right)s\right]\left(A + 2p\log(s)\right)^3\ . \tag{3.32}$$

Analyzing the behaviour of this integral directly is hampered by the fact that we are exploring the singular region where the two endpoints begin to coincide. We will therefore perform a change of variable that makes the region of integration independent of t. The transformation

$$s = \frac{1}{2}\left[\mathrm{e}^{-\hat{b}} + \mathrm{e}^{-\hat{c}} - \left(\mathrm{e}^{-\hat{c}} - \mathrm{e}^{-\hat{b}}\right)\cos(\theta)\right] \tag{3.33}$$

maps the integration domain onto $\theta \in [0,\pi]$. It also eliminates the square root in the denominator of the integrand in $\mathcal{I}$ and we arrive at

$$\mathcal{I} = \frac{c^3}{6} - \frac{1}{48\pi A^3\sqrt{1-\mathrm{e}^{-\hat{c}}}}\int_0^{\pi}\frac{\mathrm{d}\theta}{\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} - 2\ \mathrm{e}^{\hat{b}+\hat{c}} + \left(\mathrm{e}^{\hat{c}} - \mathrm{e}^{\hat{b}}\right)\cos(\theta)}$$
$$\times\frac{1}{\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} + \left(\mathrm{e}^{\hat{c}} - \mathrm{e}^{\hat{b}}\right)\cos(\theta)}\left[2\ \mathrm{e}^{\hat{b}}\sqrt{1-\mathrm{e}^{-\hat{b}}}\left(\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} - \left(\mathrm{e}^{\hat{b}} - \mathrm{e}^{\hat{c}}\right)\cos(\theta)\right)\right.$$
$$-\ \mathrm{e}^{(\hat{b}+\hat{c})/2}\sqrt{1-\mathrm{e}^{-\hat{c}}}\,p\left(\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} - 2\ \mathrm{e}^{\hat{b}+\hat{c}} + \left(\mathrm{e}^{\hat{c}} - \mathrm{e}^{\hat{b}}\right)\cos(\theta)\right)$$
$$+\left(\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} + \left(\mathrm{e}^{\hat{c}} - \mathrm{e}^{\hat{b}}\right)\cos(\theta)\right)$$
$$\left.\times\left(2\ \mathrm{e}^{\hat{b}+\hat{c}}\sqrt{1-\mathrm{e}^{-\hat{b}}} + \frac{\sqrt{1-\mathrm{e}^{-\hat{c}}}\,p}{2}\left(\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} - 2\ \mathrm{e}^{\hat{b}+\hat{c}} + \left(\mathrm{e}^{\hat{c}} - \mathrm{e}^{\hat{b}}\right)\cos(\theta)\right)\right)\right]$$
$$\times\left[A + 2p\log\left(\frac{\mathrm{e}^{-\hat{b}-\hat{c}}\left(\mathrm{e}^{\hat{b}} + \mathrm{e}^{\hat{c}} - \left(\mathrm{e}^{\hat{b}} - \mathrm{e}^{\hat{c}}\right)\cos(\theta)\right)}{2}\right)\right]^3\ . \tag{3.34}$$

Expanding $\mathcal{I}$ as a series in e^{-t} and integrating over θ, we arrive finally at

$$\frac{\partial \mathcal{F}_0(A\,,\,\hat{t}\,;\,p)}{\partial A}=\frac{\left(1-\mathrm{e}^{-2\,\hat{t}}\right)^2(p-2)^2}{8\,t^2}\,\sum_{n=1}^{\infty}\,\mathrm{e}^{-n\,t}\,G_n\big(t\,,\,\hat{t}\,;\,p\big)\ , \tag{3.35}$$

where the first three contributions are given by

$$\begin{aligned}
G_1 &= 1\ ,\\
G_2 &= \frac{\mathrm{e}^{-4\,\hat{t}}}{4}\,\Big[1+2\,p\,(2+p)+\,\mathrm{e}^{4\,\hat{t}}\,\big(1+2\,(p-2)\,p\big)+2\,\,\mathrm{e}^{2\,\hat{t}}\,\big(1-2\,p^2\big)\Big]\\
&\quad-\frac{\mathrm{e}^{-4\,\hat{t}}}{2\,t}\,\Big[\big(1-\,\mathrm{e}^{2\,\hat{t}}\big)\,\big(1+\,\mathrm{e}^{2\,\hat{t}}\,(p-1)-p\big)\,(p-2)\Big]\ ,\\
G_3 &= \frac{1}{9}-p+\frac{5}{2}\,p^2-2\,p^3+\frac{1}{2}\,p^4+\,\mathrm{e}^{-6\,\hat{t}}\,\left(\frac{2}{9}+p-p^2-4\,p^3-2\,p^4\right)\\
&\quad+\,\mathrm{e}^{-2\,\hat{t}}\,\left(\frac{2}{9}-p-p^2+4\,p^3-2\,p^4\right)+\,\mathrm{e}^{-8\,\hat{t}}\,\left(\frac{1}{9}+p+\frac{5}{2}\,p^2+2\,p^3+\frac{1}{2}\,p^4\right)\\
&\quad+\,\mathrm{e}^{-4\,\hat{t}}\left(\frac{1}{3}-3\,p^2+3\,p^4\right)-\frac{1}{t}\,\Big[\frac{2}{3}-\frac{11}{3}\,p+\frac{17}{3}\,p^2-\frac{10}{3}\,p^3+\frac{2}{3}\,p^4\\
&\quad+\,\mathrm{e}^{-4\,\hat{t}}\,\big(4\,p-2\,p^2-8\,p^3+4\,p^4\big)-\,\mathrm{e}^{-8\,\hat{t}}\,\left(\frac{2}{3}+3\,p+\frac{7}{3}\,p^2-\frac{2}{3}\,p^3-\frac{2}{3}\,p^4\right)\\
&\quad+\mathrm{e}^{-2\hat{t}}\,\left(\frac{2}{3}+p-\frac{26}{3}p^2+\frac{28}{3}p^3-\frac{8p^4}{3}\right)-\,\mathrm{e}^{-6\hat{t}}\left(\frac{2}{3}-\frac{5}{3}p-\frac{22}{3}\,p^2-\frac{4}{3}\,p^3+\frac{8}{3}p^4\right)\Big]\,.
\end{aligned} \tag{3.36}$$

To obtain the free energy we now have to integrate over the area A. For this, we express p and t in terms of A and $\hat{t}$ as $p=\frac{A}{2\,\hat{t}}$ and $t=\frac{A}{2}-2\,\hat{t}$ to write the expansion (3.35) in terms of

$$\Gamma_n\big(\hat{t}\,;\,A\big):=\frac{1}{8\,\hat{t}^{\,2}}\,\left(1-\,\mathrm{e}^{-2\,\hat{t}}\right)^2\,\,\mathrm{e}^{-n\left(\frac{A}{2}-2\,\hat{t}\right)}\,G_n\left(\frac{A}{2}-2\,\hat{t}\,,\,\hat{t}\,;\,\frac{A}{2\,\hat{t}}\right)\ . \tag{3.37}$$

For the first three terms one has

$$\begin{aligned}
\Gamma_1 &= \frac{\mathrm{e}^{-\frac{A}{2}-2\,\hat{t}}\,\big(1-\,\mathrm{e}^{2\,\hat{t}}\big)^2}{8\,\hat{t}^{\,2}}\ ,\\
\Gamma_2 &= \frac{\mathrm{e}^{-A-4\,\hat{t}}\,\big(1-\,\mathrm{e}^{2\,\hat{t}}\big)^2}{64\,\hat{t}^{\,4}}\,\Big[A^2\left(1-\,\mathrm{e}^{2\,\hat{t}}\right)^2-2\,A\left(1-\,\mathrm{e}^{2\,\hat{t}}\right)\,\left(1-2\,\hat{t}-\,\mathrm{e}^{2\,\hat{t}}\,\big(1+2\,\hat{t}\,\big)\right)\\
&\quad-4\,\left(1+\,\mathrm{e}^{2\,\hat{t}}\right)\,\hat{t}\,\left(2-\hat{t}-\,\mathrm{e}^{2\,\hat{t}}\,\big(2+\hat{t}\,\big)\right)\Big]\ ,\\
\Gamma_3 &= \frac{\mathrm{e}^{-\frac{3}{2}\,(A+4\,\hat{t}\,)}\,\big(1-\,\mathrm{e}^{2\,\hat{t}}\big)^2}{2304\,\hat{t}^{\,6}}\,\Big[9\,A^4\left(1-\,\mathrm{e}^{2\,\hat{t}}\right)^4\\
&\quad-24\,A^3\left(1-\,\mathrm{e}^{2\,\hat{t}}\right)^3\,\left(1-3\,\hat{t}-\,\mathrm{e}^{2\,\hat{t}}\,\big(1+3\,\hat{t}\,\big)\right)\\
&\quad+36\,A^2\left(1-\,\mathrm{e}^{2\,\hat{t}}\right)^2\,\hat{t}\,\left(4-5\,\hat{t}-8\,\hat{t}\,\,\mathrm{e}^{2\,\hat{t}}-\,\mathrm{e}^{4\,\hat{t}}\,\big(4+5\,\hat{t}\,\big)\right)\\
&\quad-48\,A\left(1-\,\mathrm{e}^{\hat{t}}\right)\,\hat{t}^{\,2}\,\left(5+3\,\,\mathrm{e}^{2\,\hat{t}}\,\big(1-2\,\hat{t}\,\big)-3\,\hat{t}-3\,\,\mathrm{e}^{4\,\hat{t}}\,\big(1+2\,\hat{t}\,\big)-\,\mathrm{e}^{6\,\hat{t}}\,\big(5+3\,\hat{t}\,\big)\right)
\end{aligned}$$

$$-32\,\hat{t}^{3}\,\Big(1+\,\mathrm{e}^{\,2\,\hat{t}}+\,\mathrm{e}^{\,4\,\hat{t}}\Big)\,\Big(3-\hat{t}-\hat{t}\;\mathrm{e}^{\,2\,\hat{t}}-\,\mathrm{e}^{\,4\,\hat{t}}\,\big(3+2\,\hat{t}\,\big)\Big)\Big]\;. \tag{3.38}$$

These expressions are now easily integrated over A and we arrive at the free energy components

$$F_n\big(\hat{t}\,;\,p\big) := 4\hat{t}^2\,\Big(1-\,\mathrm{e}^{\,-2\,\hat{t}}\Big)^{-2}\;\;\mathrm{e}^{\,n\left(\frac{A}{2}-2\,\hat{t}\right)}\;\;\Big[\,\int \mathrm{d}A\;\Gamma_n\big(\hat{t}\,;\,A\big)\Big]\Big|_{A=2\,p\,\hat{t}}\;. \tag{3.39}$$

For the first three contributions we find

$$\begin{aligned}
F_1 &= 1\;,\\
F_2 &= \frac{\mathrm{e}^{\,-4\,\hat{t}}}{8}\,\Big[1+4\,p+2\,p^2+\,\mathrm{e}^{\,2\,\hat{t}}\,\big(2-4\,p^2\big)+\,\mathrm{e}^{\,4\,\hat{t}}\,\big(1-4\,p+2\,p^2\big)\Big]\;,\\
F_3 &= \frac{\mathrm{e}^{\,-8\,\hat{t}}}{54}\,\Big[2+18\,p+45\,p^2+36\,p^3+9\,p^4+6\;\,\mathrm{e}^{\,4\,\hat{t}}\,\big(1-9\,p^2+9\,p^4\big)\\
&\quad+\;\,\mathrm{e}^{\,8\,\hat{t}}\,\big(2-18\,p+45\,p^2-36\,p^3+9\,p^4\big)+2\;\,\mathrm{e}^{\,6\,\hat{t}}\,\big(2-9\,p-9\,p^2+36\,p^3-18\,p^4\big)\\
&\quad+2\;\,\mathrm{e}^{\,2\,\hat{t}}\,\big(2+9\,p-9\,p^2-36\,p^3-18\,p^4\big)\Big]\;.
\end{aligned} \tag{3.40}$$

These expressions coincide with the ones obtained in (2.26) for the topological string theory on $\hat{X}_p$.

4. Analytic properties of the topological string perturbation series

Motivated by the saddle-point analysis of the phase structure of the chiral gauge theory, in this section we shall investigate the convergence properties of the perturbative topological string expansion of the partition function. In ordinary chiral Yang-Mills theory on S^2 the large N phase transition can be analysed from the string theory perspective [40]. Analytic and numerical results indicate that the string perturbation series has a finite radius of convergence which coincides with the critical points. In the Gross-Taylor string language, the phase transition is driven by the entropy of branch point singularities of the string covering maps [40]. Here we will perform an analogous investigation for the q-deformed chiral gauge theory and show that a similar picture emerges, thereby supporting the results of the previous section.

4.1 Genus zero

Recall that the genus zero free energy has the form

$$\mathcal{F}_0\big(t\,;\,p\big) = \sum_{n=1}^{\infty}\;\mathrm{e}^{\,-n\,t}\;\sum_{k=0}^{2n}\;\mathrm{e}^{\,-\frac{2\,k\,t}{p-2}}\;\mathrm{N}^0{}_{n,k}(p) \tag{4.1}$$

where $\mathrm{N}^0{}_{n,k}(p)$ is a polynomial in p of degree $2n-2$. These polynomials are given in eq. (2.26) for the first six degrees, and generally they can be parameterized as

$$\mathrm{N}^0{}_{n,k}(p) = \sum_{m=0}^{2n-2}\;\mathrm{C}^0{}_m(n,k)\;p^m\;. \tag{4.2}$$

To study the convergence properties of the series (4.1), we investigate its asymptotic large n behaviour. The dominant contributions from the genus zero Gromov-Witten invariants (4.2) are the highest degree monomials

$$\lim_{n\to\infty} \mathrm{N}^0{}_{n,k}(p) = \mathrm{C}^0{}_{2n-2}(n,k)\ p^{2n-2}\ . \tag{4.3}$$

We are thereby led to analyse the series

$$\mathcal{F}_0^\infty\big(t\,;\,p\big) = \sum_{n=1}^\infty \mathrm{e}^{\,-n\,t}\, p^{2n-2} \sum_{k=0}^{2n} \mathrm{C}^0{}_{2n-2}(n,k)\ \mathrm{e}^{\,-\frac{2\,k\,t}{p-2}}\ . \tag{4.4}$$

From a direct inspection of the first six orders in eq. (2.26), we conjecture that the combinatorial coefficients in (4.3) are given by

$$\mathrm{C}^0{}_{2n-2}(n,k) = (-1)^k\,\frac{n^{n-3}}{n!}\binom{2n}{k}\ . \tag{4.5}$$

We will motivate this conjecture below. It implies that the sum over k in (4.4) can be carried out explicitly and the expansion written as

$$\mathcal{F}_0^\infty\big(t\,;\,p\big) = \sum_{n=1}^\infty \frac{n^{n-3}}{n!}\,p^{2n-2}\left(1-\mathrm{e}^{\,-\frac{2\,t}{p-2}}\right)^{2n}\ \mathrm{e}^{\,-n\,t}\ . \tag{4.6}$$

The radius of convergence of this series is easily determined. By substituting the Stirling approximation

$$n!\ \stackrel{n\to\infty}{\longrightarrow}\ \sqrt{2\pi\,n}\ n^n\ \mathrm{e}^{\,-n} \tag{4.7}$$

we find that the large n behaviour of the free energy converges for

$$p^2\left(1-\mathrm{e}^{\,-\frac{2\,t}{p-2}}\right)^2\ \mathrm{e}^{\,-(t-1)} \le 1\ . \tag{4.8}$$

We can perform a simple check of this bound by looking at the limit in which the q-deformed gauge theory reduces to ordinary Yang-Mills theory. Rewriting eq. (4.8) in terms of the QCD_2 area parameter $A = N\,g_s p$ gives

$$p^2\left(1-\mathrm{e}^{\,-A/p}\right)^2\ \mathrm{e}^{\,-\left(\frac{A}{2}-\frac{A}{p}-1\right)} \le 1\ , \tag{4.9}$$

and in the limit $p\to\infty$ with A fixed we find

$$A^2\ \mathrm{e}^{\,-\left(\frac{A}{2}-1\right)} \le 1\ . \tag{4.10}$$

This coincides with the convergence bound derived in [40].

To understand better the approximation (4.3) and the extrapolation of the coefficients (4.5) to all orders, we need to take a closer look at the mechanism by which we recover the chiral QCD_2 string perturbation series from eq. (4.1). For this, let us reinsert the string coupling constant g_s as in eq. (2.67) to write

$$\mathcal{F}_0(A;p) = \frac{N^2p^2}{A^2}\sum_{n=1}^\infty \mathrm{e}^{\,-n\,A/2}\sum_{k=0}^{2n} \mathrm{e}^{\,(n-k)\,A/p}\ \mathrm{N}^0{}_{n,k}(p) \tag{4.11}$$

and expand the free energy in the limit $p \to \infty$ as

$$\mathcal{F}_0(A;p) = \frac{N^2 p^2}{A^2} \sum_{n=1}^{\infty} \mathrm{e}^{-n\,A/2} \sum_{k=0}^{2n} \sum_{m=0}^{2n-2} \mathrm{C}^0{}_m(n,k)\ p^m \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{A}{p}\,(n-k) \right)^l . \tag{4.12}$$

In the undeformed limit, the only contribution to the leading area term A^{2n-2} comes from the combination (4.3). The assumption that eq. (4.5) is valid at all orders implies that the coefficient of A^{2n-2} is given by

$$\sum_{k=0}^{2n} \frac{(n-k)^{2n}}{(2n)!}\ \mathrm{C}^0{}_{2n-2}(n,k) = \sum_{k=0}^{2n} \frac{(-1)^k}{(2n)!}\,(n-k)^{2n}\ \frac{n^{n-3}}{n!} \binom{2n}{k} = \frac{n^{n-3}}{n!} . \tag{4.13}$$

In [41] it was shown that this is exactly the pertinent coefficient of A^{2n-2}. It is equal to the number of (topological classes of) holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ with $2n-2$ simple branch point singularities. In the given limit the free energy (4.12) thus reduces to

$$\Phi_0^\infty(A) = N^2 \sum_{n=1}^{\infty} \frac{n^{n-3}}{n!}\ A^{2n-2}\ \mathrm{e}^{-n\,A/2} . \tag{4.14}$$

In the Gross-Taylor string description, this is the contribution to the chiral QCD_2 free energy coming from covering maps with only branch points and no Ω-point singularities. Our conjecture (4.5) thereby passes a highly non-trivial check.

Subleading terms in the limit $p \to \infty$ with A fixed must then reproduce the full undeformed chiral free energy. In ordinary chiral Yang-Mills theory, the upper radius of convergence of the string perturbation series (4.14) can be estimated to be $A_{\mathrm{c}}^+ \simeq 11.9$, while the one evaluated from the saddle-point equation for the full chiral free energy is approximately 10.189 [41]. The string estimate gives a value which is 0.2 times larger than the actual one. In the q-deformed gauge theory one can also compare the result coming from an estimate of the radius of convergence of the string perturbation series with the numerical evaluation of the saddle-point equation from section 3.3, and with the exact critical point of the full coupled gauge theory. The results are summarized in table 1. We see that the difference between the non-chiral and chiral critical normalized couplings $\hat{t}_{\mathrm{nc}}(p) = 2\,t_{\mathrm{nc}}(p)/(p-2)$ and $\hat{t}_{\mathrm{c}}^{\,\mathrm{num}}(p) = 2\,t_{\mathrm{c}}^{\,\mathrm{num}}(p)/(p-2)$ is roughly constant throughout the range of p and of the order $2-3\times10^{-3}$. The difference between the string determined and numerically determined normalized critical couplings is roughly 20% for $p>3$, and then starts to grow out of control for smaller values of p. We have also included non-integer values of p to illustrate that $\hat{t}_{\mathrm{c}}^{\,\mathrm{num}}(p)$ and $\hat{t}_{\mathrm{nc}}(p)$ both diverge with the same values as $p \to 2$, reflecting the absence of a large N phase transition for $p \le 2$.

It is interesting to note that in the $k=0$ sector of the chiral theory where the fiber D-branes are neglected, the closed topological string amplitude on X_p also has a finite radius of convergence that can be estimated in the same way. The $k=0$ contribution to the chiral free energy (4.4) is given by $\sum_{n\ge1} \frac{n^{n-3}}{n!}\,p^{2n-2}\ \mathrm{e}^{-n\,t}$, which converges for

$$p^2\ \mathrm{e}^{-(t-1)} \le 1 . \tag{4.15}$$

This is very similar to the chiral QCD_2 result (4.10).

p	$\hat{t}_{\rm nc}(p) = \frac{2\,t_{\rm nc}(p)}{p-2}$	$\hat{t}_{\rm c}^{\rm num}(p) = \frac{2\,t_{\rm c}^{\rm num}(p)}{p-2}$	$\hat{t}_{\rm c}^{\rm str}(p) = \frac{2\,t_{\rm c}^{\rm str}(p)}{p-2}$
1000	0.99×10^{-2}	1.02×10^{-2}	1.19×10^{-2}
300	3.29×10^{-2}	3.40×10^{-2}	3.97×10^{-2}
100	0.99×10^{-1}	1.02×10^{-1}	1.19×10^{-1}
30	3.30×10^{-1}	3.40×10^{-1}	3.98×10^{-1}
10	1.00	1.04	1.23
7	1.46	1.51	1.81
5	2.12	2.19	2.72
3	4.16	4.28	6.39
2.1	10.9	11.1	49.7
2.01	19.5	19.8	479
2.001	28.6	28.9	4770

Table 1: The non-chiral critical coupling constant $\hat{t}_{\rm nc}(p)$ is obtained from the full coupled gauge theory [34, 38, 39]. The chiral critical coupling $\hat{t}_{\rm c}^{\rm num}(p)$ is the larger of the two critical points obtained from the two endpoint equations of the previous section *plus* the additional equation $\rho'(c) = 0$ signalling the boundary of the physical region. The critical point $\hat{t}_{\rm c}^{\rm str}(p)$ is evaluated from the topological string perturbation series as the larger value of the two solutions to the estimate for the radius of convergence. We have normalized the variables with a factor $2/(p-2)$ because, with this choice, the equation makes sense for any value of p.

4.2 Genus one

Before moving on to discuss the implications of these results, it is natural to ask whether or not the finite radius of convergence of the perturbative expansion of the free energy is an artifact of the genus zero approximation to the full string theory. We will now show that this behaviour persists at higher genera by computing the radius of convergence of the genus one partition function. Its generic form is similar to that of the genus zero case and is given by

$$\mathcal{F}_1\big(t\,;\,p\big) = \sum_{n=1}^{\infty}\ {\rm e}^{\,-n\,t}\ \sum_{k=0}^{2n}\ {\rm e}^{\,-\frac{2\,k\,t}{p-2}}\ {\rm N}^1{}_{n,k}(p)\ , \tag{4.16}$$

where the genus one Gromov-Witten invariants ${\rm N}^1{}_{n,k}(p)$ are polynomials in the variable p of degree $2n$. Explicit forms for these polynomials are given for the first five degrees in appendix B, and in general they may be parameterized as

$${\rm N}^1{}_{n,k}(p) = \sum_{m=0}^{2n}\ {\rm C}^1{}_m(n,k)\ p^m\ . \tag{4.17}$$

At large n the dominant contributions again come from the maximal degree monomials

$$\lim_{n\to\infty}\ {\rm N}^1{}_{n,k}(p) = {\rm C}^1{}_{2n}(n,k)\ p^{2n}\ , \tag{4.18}$$

and the radius of convergence can be estimated from the infinite series

$$\mathcal{F}_1^{\infty}\big(t\,;\,p\big) = \sum_{n=1}^{\infty}\ \mathrm{e}^{\,-n\,t}\,p^{2n}\ \sum_{k=0}^{2n}\mathrm{C}^1{}_{2n}(n,k)\ \ \mathrm{e}^{\,-\frac{2\,k\,t}{p-2}}\ . \tag{4.19}$$

Analogously to the genus zero case, by inspection of the first five orders of the series we conjecture that

$$\mathrm{C}^1{}_{2n}(n,k) = (-1)^k\ \frac{R(n)}{(2n)!}\begin{pmatrix}2n\\ k\end{pmatrix} \tag{4.20}$$

where

$$R(n) = \frac{(2n)!}{24\,n!}\left(n^n - n^{n-1} - \sum_{m=2}^{n}\,(m-2)!\begin{pmatrix}n\\ m\end{pmatrix}n^{n-m}\right) =:\,(2n)!\ H(n)\ . \tag{4.21}$$

In the Gross-Taylor description, the combinatorial coefficient $R(n)$ counts the number of branched coverings of a sphere by a torus with simple ramification corresponding to the trivial partition $\vec{\mu} = (1^n)$ [55], i.e. with no Ω-point singularities. We can now perform the sum over k explicitly in (4.19) to get

$$\mathcal{F}_1^{\infty}\big(t\,;\,p\big) = \sum_{n=1}^{\infty}\,H(n)\,p^{2n}\left(1-\,\mathrm{e}^{\,-\frac{2\,t}{p-2}}\right)^{2n}\ \ \mathrm{e}^{\,-n\,t}\ . \tag{4.22}$$

The radius of convergence of this series can be easily evaluated by substituting the asymptotic behaviour (see appendix B)

$$\lim_{n\to\infty}\ H(n) = \frac{\mathrm{e}^{\,n}}{48n}\ . \tag{4.23}$$

The free energy (4.22) thereby converges when

$$p^2\left(1-\,\mathrm{e}^{\,-\frac{2\,t}{p-2}}\right)^2\ \ \mathrm{e}^{\,-(t-1)}\leq 1\ . \tag{4.24}$$

This is identical to the convergence bound obtained at genus zero in (4.8), suggesting that the divergence of the topological string perturbation series is a generic property of the partition function which holds to all orders in the genus expansion.

4.3 Phase transition as a Hagedorn transition

The results of this section indicate that the perturbative expansion of the topological string theory under examination has a finite radius of convergence. In particular, even the closed topological string amplitude on X_p exhibits this same behaviour according to eq. (4.15). We immediately see that this conclusion is incorrect for the cases $p = 1, 2$. For $p = 1$ we cannot disregard the lower powers of p in estimating the asymptotic behaviour, and the perturbative contribution to the free energy sums to a polylogarithm function giving

$$F^0{}_{\substack{0,0\\0,0}}(t;1) = \mathrm{n}^0{}_1(1)\ \mathrm{Li}_3\big(\,\mathrm{e}^{\,-t}\,\big) \tag{4.25}$$

at genus zero [44], as ${\rm n}^0{}_1(1)$ is the only non-vanishing Gopakumar-Vafa invariant in this case. The case $p = 2$ is more subtle to handle but since again it leads to just one non-vanishing integer invariant ${\rm n}^0{}_1(2)$, the same formula holds. The cases $p = 1, 2$ are special both from the point of view of the saddle-point analysis performed in this paper and in the full partition function [34, 38, 39] constructed through coupled SU(N) representations. Instead, for $p \geq 3$ we expect our analysis to be correct. The Gopakumar-Vafa invariants in this case do not seem to vanish after any finite degree, and the topological string amplitude at genus zero is organized as an infinite series

$$F^0{}_{\substack{0,0\\0,0}}(t;p) = \sum_{k=1}^{\infty} {\rm n}^0{}_k(p)\ {\rm Li}_3\big({\rm e}^{-k\,t}\big) \tag{4.26}$$

that can diverge at some critical value $t = t_{\rm c}(p)$. From eq. (4.15) we see that the string expansion should diverge for $t < t_{\rm c}(p)$ with

$$t_{\rm c}(p) = 1 + 2\,\log(p)\ . \tag{4.27}$$

One can now speculate on the meaning of this divergence in the context of closed topological string theory, and wonder if there is any method of analytical continuation that would allow the definition of the theory below the critical radius. Because the topological string partition function counts BPS bound states of D0-D2 branes in Type IIA string theory on X_p weighted by their BPS energy [49, 50], it is tempting to propose an interpretation of the divergence as a sort of Hagedorn transition. The number of BPS states, which are counted by the Gopakumar-Vafa invariants, grows exponentially leading to a Hagedorn-like behaviour.

Coming back to the chiral string theory studied in this paper we see that the string perturbation series converges in two different regions of ${\rm Re}(t) > 0$, in contrast to the non-chiral string theory. This mimicks the behaviour obtained from the saddle-point analysis in section 3.3 (see figure 9). We have already noted that the larger critical point derived from eq. (4.8) is in good agreement with the higher phase transition point. The region of small t instead does not seem to be well described by the string estimate, as the big difference between the numerical and the analytical evaluations of the second critical point shows.

5. Conclusions

In this paper we have analysed the large N limit of the chiral sector of q-deformed Yang-Mills theory on the sphere and its relation with the topological string, as a part of the project initiated in [34]. We have found a rich picture where a toric structure and the geometry of its rationally embedded curves emerge from the chiral sector of the gauge theory. Confirming the proposal of [28, 34], the strong-coupling phase of q-deformed chiral Yang-Mills theory is a topological string theory and its counting of branched covering maps is related to the counting of worldsheet instantons in the topological sigma-model. The phase structure of the gauge theory is similar to the familiar Crescimanno-Taylor picture, to which it is smoothly connected in the undeformed limit. The presence of a phase transition is also confirmed by the finite radius of convergence of the perturbative string expansion.

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In the strong coupling phase the chiral deformed theory is related to an emerging toric geometry through the topological vertex formalism of [42]. The gauge theory thus captures the Calabi-Yau geometry with the appropriate D-brane insertions that are relevant for the counting of the black hole microstates in the effective four dimensional supergravity theory. In this setup the "distance" between the fiber D-branes and the base sphere plays the role of a geometric Kähler modulus. Remarkably, the large N chiral q-deformed theory can be used efficiently to compute Gromov-Witten and Gopakumar-Vafa invariants of the toric geometry. On the other hand, the Gross-Taylor expansion of QCD_2 is recovered in a suitable double scaling limit. This expansion is known to compute Hurwitz numbers, the combinatorics of branched coverings of the sphere. This connection is summarized by the explicit localization formula for the Gromov-Witten invariants. Moreover, this relation can be exploited to study the analytic properties of the topological string perturbation series. The existence of a finite radius of convergence reflects the phase structure that we derived from a numerical analysis of the matrix model. Because the topological string is connected to the counting of BPS states, the divergence of its perturbation series could be physically related to a Hagedorn transition. A possibly related phase transition has been found recently in [56], where the topological $\mathcal{N} = 4$ gauge theory is studied which closely resembles the gauge theory defined on the N D4-branes that localizes to the q-deformed Yang-Mills theory.

Together with the results in [34] we find a satisfactory and consistent confirmation of the ideas presented in [28]. The deformed Yang-Mills theory that computes the BPS degeneracies of four-dimensional black holes is deeply related to the geometrical invariants of the relevant Calabi-Yau threefold. On the other hand, these are the objects underlying the topological string amplitudes that compute the four-dimensional effective field theory F-terms. The phase structure of the gauge theory is reflected in the topological string amplitude in a very precise fashion. The topological string theory also provides an explicit realization of the Gross-Taylor string expansion of the q-deformed Yang-Mills theory.

In principle, the techniques used in this paper can be extended to compute Gromov-Witten invariants for the more general geometries of [28]. Similar issues were already addressed in [33]. The chiral expansion of q-deformed Yang-Mills theory could in general lead to a better understanding of the relevant Calabi-Yau threefolds.

It would be fruitful to understand better the implications of the phase transition for black holes physics. In the coupled expansion we encountered a third-order phase transition of Gross-Witten type. Recently it has been proposed [57, 58] that such behaviour could be related to a change of regime from a macroscopic black hole to a perturbative string state, in the spirit of the Horowitz-Polchinski transition [59]. On the other hand the exponential growth in the density of BPS states observed here should suggest some type of Hagedorn behaviour. It would be interesting to find a string dual exhibiting the same growth of states at the perturbative level, as in the correspondence between heterotic strings on $\mathbb{T}^6$ and Type IIA strings on $K3 \times \mathbb{T}^2$ [60], and to understand the phase transition there as done in [15].

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A. Degree zero partition function

In this appendix we will derive the contribution $\mathsf{Z}_0(q)$ of constant maps to the topological string partition function by computing the normalization of the q-deformed Yang-Mills partition function. For this, we consider the quantity

$$\begin{aligned} S_{00}(q,N) &:= \prod_{1\le i<j\le N} \Big[q^{(j-i)/2} - q^{-(j-i)/2}\Big] \\ &= \exp\Big[-\sum_{1\le i<j\le N}\Big(\frac{j-i}{2}\log(q) - \log\big(1-q^{j-i}\big)\Big)\Big] \ , \end{aligned} \tag{A.1}$$

where the second line holds up to an irrelevant phase factor. The first term in the exponential can be easily summed according to

$$\sum_{1\le i<j\le N} \frac{j-i}{2} = \frac{1}{2}\sum_{i=1}^{N-1}\Big(\frac{N\,(N+1)}{2} - \frac{i\,(i+1)}{2} + i^2 - N\,i\Big) = \frac{N^2\,(N-1)}{12} \ . \tag{A.2}$$

In the second term, since $q = \mathrm{e}^{-g_s} \in (0,1)$ and $j-i>0$ we can expand the logarithm in powers of q to obtain

$$\begin{aligned} \sum_{1\le i<j\le N} \log\big(1-q^{j-i}\big) &= -\sum_{m=1}^{\infty}\sum_{i=1}^{N-1} \frac{q^{-i\,m}}{m}\,\frac{q^{m\,(N+1)} - q^{m\,(i+1)}}{q^m-1} \\ &= -\sum_{m=1}^{\infty}\Big(\frac{N\,q^m}{m\,(1-q^m)} - \frac{q^m}{m\,(1-q^m)^2} + \frac{q^{m\,(N+1)}}{m\,(1-q^m)^2}\Big) \ . \end{aligned} \tag{A.3}$$

The first two terms in (A.4) can be rewritten in more familiar forms by expanding again in powers of q to get

$$-\sum_{m=1}^{\infty}\frac{N\,q^m}{m\,(1-q^m)} = -N\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q^{m\,n}}{m} = N\,\log\big(\eta(q)\big) - \frac{N}{24}\,\log(q) \tag{A.4}$$

where

$$\eta(q) = q^{1/24}\,\prod_{n=1}^{\infty}(1-q^n) \tag{A.5}$$

is the Dedekind function, and similarly

$$\sum_{m=1}^{\infty}\frac{q^m}{m\,(1-q^m)^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n}{m}\,q^{m\,n} = \log\big(M(q)\big) \tag{A.6}$$

where

$$M(q) = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n} \tag{A.7}$$

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is the McMahon function. The series expansion of (A.7) is the generating function $\sum_{\mathcal{Y}} q^{|\mathcal{Y}|}$ for plane partitions $\mathcal{Y}$ which can be used to define the topological vertex of section 2.3.

Collecting everything together we arrive at

$$\begin{aligned}\log\big(S_{00}(q,N)\big) &= -\frac{1}{24}\left(2N^2\,(N-1)+N\right)\,\log(q)+N\,\log\big(\eta(q)\big)\\ &\quad+\log\big(M(q)\big)+\log\big(N_0(q,Q)\big)\ ,\end{aligned}\tag{A.8}$$

where we have introduced the function

$$N_0(q,Q)=\exp\Big(-\sum_{n=1}^{\infty}B_n(q)\ Q^n\Big)\tag{A.9}$$

with $Q:=q^N\ =\ \mathrm{e}^{\,-g_sN}$ and

$$B_n(q)=\frac{q^n}{n\,(1-q^n)^2}\ .\tag{A.10}$$

The function $N_0(q,Q)$ encodes the nonperturbative corrections to the degree zero map contribution, which are immaterial in the large N limit. We may therefore factor it out of the partition function, after which we arrive at the result (2.48) reported in the main text.

The corresponding free energy

$$\mathsf{F}_0(q)=\lim_{N\to\infty}\ \log\left(\frac{S_{00}(q,N)}{N_0(q,Q)}\right)\tag{A.11}$$

admits a genus expansion $\mathsf{F}_0=\sum_{g\geq0}g_s^{2g-2}\ {\mathsf{F}_0}^g$. Its coefficients reproduce the degree zero Gromov-Witten invariants $\hat{\mathrm{N}}^g{}_{(0,0,0)}(p)={\mathrm{N}^g}_0(p)$ which can be expressed entirely in terms of classical intersection indices and characteristic classes of the threefold X_p as

$$\begin{aligned}{\mathrm{N}^0}_0(p) &= \frac{1}{3!}\sum_{\alpha_{a_1},\alpha_{a_2},\alpha_{a_3}\in H^*(X_p,\mathbb{Z})}\ \int_{X_p}\alpha_{a_1}\wedge\alpha_{a_2}\wedge\alpha_{a_3}\ ,\\ {\mathrm{N}^1}_0(p) &= -\frac{1}{24}\int_{X_p}\tau\wedge c_2(X_p)\ ,\\ {\mathrm{N}^g}_0(p) &= \frac{(-1)^g\,|B_{2g}\,B_{2g-2}|}{4g\,(2g-2)\,(2g-2)!}\int_{X_p}\big(c_3(X_p)-c_1(X_p)\wedge c_2(X_p)\big)\end{aligned}\tag{A.12}$$

with $g\geq2$. Here $\{\alpha_a\}$ is a basis of $H^*(X_p,\mathbb{Z})$ modulo torsion, the class τ is the degree two cohomology generator and $c_n(X_p)$ is the n-th Chern class of the tangent bundle TX_p. The coefficients $B_{2n}\in\mathbb{Q}$ are the Bernoulli numbers.

B. Genus one partition function

In this appendix we collect the first five contributions to the genus one partition function. Let us parameterize the genus one free energy as

$$\mathcal{F}_1\big(t\,,\,\hat{t}\,;\,p\big)=\Big(1-\mathrm{e}^{\,-2\,\hat{t}}\Big)^2\ \sum_{n=1}^{\infty}\ \mathrm{e}^{\,-n\,t}\ T_n\big(\hat{t}\,;\,p\big)\ .\tag{B.1}$$

Then the first five coefficients are given by

$$T_1 = -\frac{1}{12} \, ,$$

$$T_2 = -\frac{\mathrm{e}^{-4\hat{t}}}{48} \Big[2 + 6p - p^2 - 4p^3 - p^4 + 2\ \mathrm{e}^{2\hat{t}} \left(2 - 5p^2 + p^4\right)$$
$$+\ \mathrm{e}^{4\hat{t}} \left(2 - 6p - p^2 + 4p^3 - p^4\right)\Big] \, ,$$

$$T_3 = -\frac{\mathrm{e}^{-8\hat{t}}}{72} \Big[2 + 14p + 19p^2 - 20p^3 - 45p^4 - 24p^5 - 4p^6$$
$$+6\ \mathrm{e}^{4\hat{t}} \left(1 - 9p^2 + 13p^4 - 4p^6\right)$$
$$+\ \mathrm{e}^{8\hat{t}} \left(2 - 14p + 19p^2 + 20p^3 - 45p^4 + 24p^5 - 4p^6\right)$$
$$+2\ \mathrm{e}^{6\hat{t}} \left(2 - 7p - 14p^2 + 34p^3 + 3p^4 - 24p^5 + 8p^6\right)$$
$$+2\ \mathrm{e}^{2\hat{t}} \left(2 + 7p - 14p^2 - 34p^3 + 3p^4 + 24p^5 + 8p^6\right)\Big] \, ,$$

$$T_4 = -\frac{\mathrm{e}^{-12\hat{t}}}{288} \Big[6 + 70p + 227p^2 + 80p^3 - 744p^4 - 1332p^5 - 950p^6 - 312p^7 - 39p^8$$
$$+\ \mathrm{e}^{12\hat{t}} \left(6 - 70p + 227p^2 - 80p^3 - 744p^4 + 1332p^5 - 950p^6 + 312p^7 - 39p^8\right)$$
$$+2\ \mathrm{e}^{10\hat{t}} \left(6 - 44p - 12p^2 + 452p^3 - 543p^4 - 438p^5 + 1080p^6 - 624p^7 + 117p^8\right)$$
$$+2\ \mathrm{e}^{2\hat{t}} \left(6 + 44p - 12p^2 - 452p^3 - 543p^4 + 438p^5 + 1080p^6 + 624p^7 + 117p^8\right)$$
$$+4\ \mathrm{e}^{6\hat{t}} \left(6 - 118p^2 + 471p^4 - 560p^6 + 195p^8\right)$$
$$+\ \mathrm{e}^{8\hat{t}} \left(18 - 70p - 327p^2 + 832p^3 + 888p^4 - 2244p^5 - 90p^6 + 1560p^7 - 585p^8\right)$$
$$+\ \mathrm{e}^{4\hat{t}} \left(18 + 70p - 327p^2 - 832p^3 + 888p^4 + 2244p^5 - 90p^6 - 1560p^7 - 585p^8\right)\Big] \, ,$$

$$T_5 = -\frac{\mathrm{e}^{-16\hat{t}}}{1440} \Big[24 + 404p + 2182p^2 + 3660p^3 + 5865p^4 - 31604p^5 - 50870p^6$$
$$-42200p^7 - 19435p^8 - 4720p^9 - 472p^{10} +\ \mathrm{e}^{10\hat{t}} \big(96 - 404p - 3242p^2 + 8620p^3$$
$$+21470p^4 - 50176p^5 - 36880p^6 + 101800p^7 - 1540p^8 - 66080p^9 + 26432p^{10}\big)$$
$$+\ \mathrm{e}^{16\hat{t}} \big(24 - 404p + 2182p^2 - 3660p^3 - 5865p^4 + 31604p^5 - 50870p^6 + 42200p^7$$
$$-19435p^8 + 4720p^9 - 472p^{10}\big) + 2\ \mathrm{e}^{14\hat{t}} \big(24 - 278p + 397p^2 + 3990p^3 - 13475p^4$$
$$+5208p^5 + 32360p^6 - 57340p^7 + 41410p^8 - 14160p^9 + 1888p^{10}\big) + 2\ \mathrm{e}^{2\hat{t}} \big(24$$
$$+278p + 397p^2 - 3990p^3 - 13475p^4 - 5208p^5 + 32360p^6 + 57340p^7 + 41410p^8$$
$$+14160p^9 + 1888p^{10}\big) + 10\ \mathrm{e}^{8\hat{t}} \big(12 - 418p^2 + 3257p^4 - 8810p^6 + 9275p^8$$
$$-3304p^{10}\big) + 4\ \mathrm{e}^{12\hat{t}} \big(18 - 139p - 311p^2 + 3010p^3 - 1235p^4 - 13436p^5$$
$$+16770p^6 + 9180p^7 - 27055p^8 + 16520p^9 - 3304p^{10}\big) + 4\ \mathrm{e}^{4\hat{t}} \big(18 + 139p$$
$$-311p^2 - 3010p^3 - 1235p^4 + 13436p^5 + 16770p^6 - 9180p^7 - 27055p^8$$
$$-16520p^9 - 3304p^{10}\big) + 2\ \mathrm{e}^{6\hat{t}} \big(48 + 202p - 1621p^2 - 4310p^3 + 10735p^4$$
$$+25088p^5 - 18440p^6 - 50900p^7 - 770p^8 + 33040p^9 + 13216p^{10}\big)\Big] \, . \tag{B.2}$$

For the analysis of section 4.2 we require only the leading order expansion in p given by

$$\begin{aligned}\mathcal{F}_1^\infty(t\,,\,\hat{t}\,;\,p) &= \frac{1}{48}\left(1-\mathrm{e}^{\,-2\,\hat{t}}\right)^4\ \mathrm{e}^{\,-2\,t}\,p^4+\frac{1}{18}\left(1-\mathrm{e}^{\,-2\,\hat{t}}\right)^6\ \mathrm{e}^{\,-3\,t}\,p^6\\ &\quad+\frac{13}{96}\left(1-\mathrm{e}^{\,-2\,\hat{t}}\right)^8\ \mathrm{e}^{\,-4\,t}\,p^8+\frac{59}{180}\left(1-\mathrm{e}^{\,-2\,\hat{t}}\right)^{10}\ \mathrm{e}^{\,-5\,t}\,p^{10}+\cdots\\ &= \sum_{n=1}^\infty\frac{R(n)}{(2n)!}\left(1-\mathrm{e}^{\,-2\,\hat{t}}\right)^{2n}\ \mathrm{e}^{\,-n\,t}\,p^{2n}\ . \end{aligned}\tag{B.3}$$

The coefficients $R(n)$ corresponding to the different winding numbers are recognized to be

$$R(n)=\frac{(2n)!}{24\,n!}\left(n^n-n^{n-1}-\sum_{k=2}^n(k-2)!\binom{n}{k}\,n^{n-k}\right)\ . \tag{B.4}$$

They count the number of branched coverings of the sphere by a torus with simple ramification corresponding to the trivial partition (1^n).

The asymptotic behaviour of $R(n)$ for large n can be determined by means of the integral representation

$$\begin{aligned}H(n):=\frac{R(n)}{(2n)!} &= \frac{1}{24\,n!}\left(n^n-n^{n-1}-\int_0^\infty\mathrm{d}t\ \sum_{k=2}^n t^{k-2}\ \mathrm{e}^{\,-t}\binom{n}{k}\,n^{n-k}\right)\\ &= \frac{1}{24\,n!}\left(n^n-n^{n-1}-n^n\int_0^\infty\frac{\mathrm{d}t}{t^2}\ \mathrm{e}^{\,-t}\left[-1-t+\left(\frac{n+t}{n}\right)^n\right]\right)\\ &= \frac{1}{24\,n!}\left(n^n-n^{n-1}-\int_0^\infty\mathrm{d}t\ \mathrm{e}^{\,-t}\left[n^n-(n+t)^{n-1}\right]\right)\\ &= \frac{1}{24\,n!}\left(\int_0^\infty\mathrm{d}t\ \mathrm{e}^{\,-t}\,(n+t)^{n-1}-n^{n-1}\right)\ . \end{aligned}\tag{B.5}$$

This last integral can be computed explicitly, giving a closed form for these combinatorial numbers in terms of the incomplete gamma-function as

$$H(n)=\frac{1}{24\,n!}\left(\mathrm{e}^{\,n}\,\Gamma(n,n)-n^{n-1}\right)=\frac{1}{24\,n!}\left(\frac{\mathrm{e}^{\,n}\,\Gamma(n+1,n)}{n}-2\,n^{n-1}\right)\ . \tag{B.6}$$

By using the asymptotic large n expansion

$$\Gamma(n+1,n)=\mathrm{e}^{\,-n}\,n^n\left(\sqrt{\frac{\pi}{2}}\,n^{1/2}+\frac{2}{3}+\frac{\sqrt{2\pi}}{24}\,n^{-1/2}+\cdots\right) \tag{B.7}$$

along with the Stirling approximation we obtain

$$\lim_{n\to\infty}H(n)=\frac{n^{n-1/2}\sqrt{\frac{\pi}{2}}}{24\,n!}\ \stackrel{n\to\infty}{\longrightarrow}\ \frac{n^{n-1/2}\sqrt{\frac{\pi}{2}}}{24\,n^{n+1/2}\sqrt{2\pi}\ \mathrm{e}^{\,-n}}=\frac{\mathrm{e}^{\,n}}{48n}\ . \tag{B.8}$$

C. Gopakumar-Vafa invariants

In this appendix we list the genus zero Gopakumar-Vafa invariants ${\rm n}^0{}_n(p) = \hat{\rm n}^0{}_{(0,n,0)}(p)$ of the threefold X_p for $n = 1, \dots, 7$, all of which are computed by the method described in section 2.5:

$$
\begin{aligned}
{\rm n}^0{}_1(p) &= (-1)^p \ , \\
{\rm n}^0{}_2(p) &= \frac{1}{8}\left(1-(-1)^p-4\,p+2\,p^2\right) \ , \\
{\rm n}^0{}_3(p) &= -(-1)^p\left(\frac{1}{3}\,p-\frac{5}{6}\,p^2+\frac{2}{3}\,p^3-\frac{1}{6}\,p^4\right) , \\
{\rm n}^0{}_4(p) &= -\left(\frac{1}{6}\,p-\frac{13}{12}\,p^2+\frac{7}{3}\,p^3-\frac{9}{4}\,p^4+p^5-\frac{1}{6}\,p^6\right) \ , \\
{\rm n}^0{}_5(p) &= -(-1)^p\left(\frac{1}{6}\,p-\frac{5}{4}\,p^2+\frac{9}{2}\,p^3-\frac{69}{8}\,p^4+\frac{55}{6}\,p^5-\frac{65}{12}\,p^6+\frac{5}{3}\,p^7-\frac{5}{24}\,p^8\right) \ , \\
{\rm n}^0{}_6(p) &= -\left(\frac{13}{120}\,p-\frac{(-1)^p}{24}\,p-\frac{313}{240}\,p^2+\frac{5\,(-1)^p}{48}\,p^2+\frac{83}{12}\,p^3-\frac{(-1)^p}{12}\,p^3-\frac{333}{16}\,p^4\right. \\
&\qquad \left.+\frac{(-1)^p}{48}\,p^4+\frac{757}{20}\,p^5-\frac{1025}{24}\,p^6+30\,p^7-\frac{51}{4}\,p^8+3\,p^9-\frac{3}{10}\,p^{10}\right) \ , \\
{\rm n}^0{}_7(p) &= -(-1)^p\left(\frac{1}{10}\,p-\frac{241}{180}\,p^2+\frac{851}{90}\,p^3-\frac{7163}{180}\,p^4+\frac{38269}{360}\,p^5-\frac{134407}{720}\,p^6\right. \\
&\qquad \left.+\frac{19747}{90}\,p^7-\frac{24941}{144}\,p^8+\frac{6517}{72}\,p^9-\frac{2401}{80}\,p^{10}+\frac{343}{60}\,p^{11}-\frac{343}{720}\,p^{12}\right) \ . \qquad \text{(C.1)}
\end{aligned}
$$

Note that for $p = 0, 1, 2$ one has ${\rm n}^0{}_n(p) = 0$ for $n \neq 1$, and by using eq. (2.58) one finds that the corresponding Gromov-Witten invariants are given by ${\rm N}^0{}_n(p) = (-1)^p/n^3$. This is the expected result for smoothly embedded contractible rational curves [51]. For $p \geq 3$ the structure changes. For example, at $p = 3$ one finds ${\rm n}^0{}_7(3) \neq 0$.

D. Chiral integrals

The saddle-point solution of the chiral q-deformed gauge theory in the large N limit requires the elementary indefinite integral

$$
\begin{aligned}
&\int \frac{{\rm d}w}{w-s}\,\frac{1}{\sqrt{(w-{\rm e}^{\,c'})(w-{\rm e}^{\,b'})}} \qquad \text{(D.1)}\\
&\quad = -\frac{1}{\sqrt{(s-{\rm e}^{\,c'})(s-{\rm e}^{\,b'})}}\,\log\left(\frac{\left(\sqrt{(w-{\rm e}^{\,b'})(s-{\rm e}^{\,c'})}+\sqrt{(s-{\rm e}^{\,b'})(w-{\rm e}^{\,c'})}\right)^2}{(s-w)\sqrt{(s-{\rm e}^{\,b'})(s-{\rm e}^{\,c'})}}\right)
\end{aligned}
$$

in the complex plane. The cuts in both s and w are taken as indicated in figure 6. Then one has

$$
\int_{-\infty}^{-\epsilon} \frac{{\rm d}w}{w(w-s)}\,\frac{1}{\sqrt{(\,{\rm e}^{\,c'}-w)(\,{\rm e}^{\,b'}-w)}} \qquad \text{(D.2)}
$$

$$
\begin{aligned}
&= \frac{1}{s}\left(\int_{-\infty}^{-\epsilon}\frac{\mathrm{d}w}{w-s}\,\frac{1}{\sqrt{(\mathrm{e}^{\,c'}-w)(\mathrm{e}^{\,b'}-w)}}-\int_{-\infty}^{-\epsilon}\frac{\mathrm{d}w}{w}\,\frac{1}{\sqrt{(\mathrm{e}^{c'}-w)(\mathrm{e}^{b'}-w)}}\right)\\
&= \frac{1}{s\sqrt{(s-\mathrm{e}^{\,b'})(s-\mathrm{e}^{\,c'})}}\log\left(\frac{\left(\mathrm{e}^{\,b'/2}\sqrt{s-\mathrm{e}^{\,c'}}+\mathrm{e}^{\,c'/2}\sqrt{s-\mathrm{e}^{\,b'}}\right)^2}{s\left(\sqrt{s-\mathrm{e}^{\,c'}}+\sqrt{s-\mathrm{e}^{\,b'}}\right)^2}\right)\\
&\qquad +\frac{\mathrm{e}^{\,-(b'+c')/2}}{s}\left[b'+c'-2\log\left(\frac{\mathrm{e}^{\,b'/2}+\mathrm{e}^{\,c'/2}}{2}\right)-\log(\epsilon)\right].
\end{aligned}
$$

The integral over the cut $[\,\mathrm{e}^{\,c'},\,\mathrm{e}^{\,d'}]$ is instead given by

$$
\begin{aligned}
&\int_{\mathrm{e}^{\,c'}}^{\mathrm{e}^{\,d'}}\frac{\mathrm{d}w}{w(w-s)}\,\frac{1}{\sqrt{(\mathrm{e}^{\,c'}-w)(\mathrm{e}^{\,b'}-w)}} \qquad\qquad (\mathrm{D}.3)\\
&\quad= \frac{1}{s}\left(\int_{\mathrm{e}^{\,c'}}^{\mathrm{e}^{\,d'}}\frac{\mathrm{d}w}{w-s}\,\frac{1}{\sqrt{(\mathrm{e}^{\,c'}-w)(\mathrm{e}^{\,b'}-w)}}-\int_{\mathrm{e}^{\,c'}}^{\mathrm{e}^{\,d'}}\frac{\mathrm{d}w}{w}\,\frac{1}{\sqrt{(\mathrm{e}^{\,c'}-w)(\mathrm{e}^{\,b'}-w)}}\right)\\
&\quad= -\frac{1}{s\sqrt{(s-\mathrm{e}^{\,c'})(s-\mathrm{e}^{\,b'})}}\log\left(\frac{\left(\sqrt{(\mathrm{e}^{\,d'}-\mathrm{e}^{\,b'})(s-\mathrm{e}^{\,c'})}+\sqrt{(s-\mathrm{e}^{\,b'})(\mathrm{e}^{\,d'}-\mathrm{e}^{\,c'})}\right)^2}{(s-\mathrm{e}^{\,d'})\,(\mathrm{e}^{\,c'}-\mathrm{e}^{\,b'})}\right)\\
&\qquad -\frac{\mathrm{e}^{\,-(b'+c')/2}}{s}\log\left(\frac{\left(\mathrm{e}^{\,c'/2}\sqrt{\mathrm{e}^{\,d'}-\mathrm{e}^{\,b'}}+\mathrm{e}^{\,b'/2}\sqrt{\mathrm{e}^{\,d'}-\mathrm{e}^{\,c'}}\right)^2}{\mathrm{e}^{\,d'}\,(\mathrm{e}^{\,c'}-\mathrm{e}^{\,b'})}\right).
\end{aligned}
$$

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